Free theories of definite descriptions based on *FDE*

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Introduction

- The majority of free logics are based on classical sentential logic, and so are the majority of free theories of DDs.
- A very natural principle governing DDs, viz., the **characterization principle** (CP), gives rise to paradoxes whose structure resembles that of well-known paradoxes, such as Russell's and Curry's paradoxes.
- Is it possible to come up with a non-trivial free theory of DDs that retains CP? In particular, would a free theory of DDs that is based on **first-degree entailment** (*FDE*) be capable of handling the aforementioned paradoxes?
  - *FDE* is a paraconsistent logic.
  - *FDE* lacks an implication connective w.r.t. which *modus ponens* is valid.
- The answer is "No". Such a theory is either trivial or must be subject to severe restrictions that end up rendering it completely useless.
- One can still come up with non-trivial theories of DDs that are based on *FDE*.

Introduction

In a series of recent papers Abilio Rodrigues and I have investigated different first-order versions of *FDE* and a few related logics:

- H. Antunes, A. Rodrigues, W. Carnielli, and M.E. Coniglio. Valuation semantics for first-order logics of evidence and truth. *Journal of Philosophical Logic* 51:1141–73, 2022.
  - Constant domain versions of *FDE*, *K*3, *LP*, and LET_F
- A. Rodrigues and H. Antunes. First-order logics of evidence and truth with constant and variable domains. *Logica Universalis* 16:419-49, 2022.
  - Constant and variable domain versions of LET_F and LET_J
- H. Antunes and A. Rodrigues. Variable domain first-order first-degree entailment and some of its children. *Studia Logica*. *Forthcoming*.
  - Variable domain versions of *FDE*, *K*3, *LP*, *N*3, *N*4, and *NP*
- H. Antunes and A. Rodrigues. On universally free Belnap-Dunn's four-valued logic and Nelson's *N*4: Technical Results. *Forthcoming*.
  - Variable domain free versions of *FDE* and *N*4

Outline

1. Free theories of definite descriptions
2. Definite description paradoxes I
3. The logic *FFDE* (in collaboration with Abilio Rodrigues)
4. Definite description paradoxes II
5. Free theories of definite descriptions based on *FDE* (ongoing work)

Free theories of definite descriptions

- Free logics allow for **empty singular terms**, i.e., terms that are not assigned an element of the quantificational domain.
  - t is not assigned any object at all (**single domain**)
  - t is assigned an object outside the domain of the quantifiers (**dual domain**)
- In a free logic the usual $\forall E$ and $\exists I$ rules are no longer valid and must be replaced by weaker versions thereof:

$$\forall x\alpha \nvDash \alpha(t/x) \qquad \forall x\alpha, \boldsymbol{E}t \vDash \alpha(t/x)$$

$$\alpha(t/x) \nvDash \exists x\alpha \qquad \alpha(t/x), \boldsymbol{E}t \vDash \exists x\alpha$$

- If the logic is **inclusive**, its $\forall I$ and $\exists E$ rules must also be modified in a similar way.
- Even though free logics are non-classical, the majority of them is **sententially classical**, in the sense that they preserve all valid inferences of sentencial classical logic.

Free theories of definite descriptions

Definite description operator

$$\iota x\alpha \;\Rightarrow\; \textit{the individual that satisfies } \alpha \Rightarrow \underline{\textit{the } \alpha}$$

(x is the only variable free in α)

- ι is a **variable binding term operator**.
- α is the **basis** of $\iota x\alpha$ and the **scope** of x.
- $\iota x\alpha$ is **proper** iff there exists one and only one individual in the domain of the quantifiers that satisfies α; $\iota x\alpha$ is **improper** otherwise.

- In a free theory...
  - $\iota x\alpha$ is a genuine singular term (≠ Russellian treatment)
  - Improper DDs are treated as empty terms

Free theories of definite descriptions

All free theories of DDs (that I am aware of) agree on **Lambert's law**:

Lambert's law

(LL) $\forall y[\ y = \imath x\alpha \ \leftrightarrow\ \forall x(\alpha \leftrightarrow x = y)\ ]$

(Lambert [15])

- **Example:**

$$\forall y[\ y = \imath xFx \ \leftrightarrow\ \forall x(Fx \leftrightarrow x = y)\ ]$$

- LL expresses that an object d in the domain of the quantifiers is (equal to) *the* α iff d is the only object in the domain of the quantifiers that satisfies α.
- LL says nothing about the behavior of improper DDs.

Free theories of definite descriptions

A few consequences of LL in **positive free logic**:

- $\boldsymbol{E}\imath x\alpha \rightarrow \imath x\alpha = \imath x\alpha$
- $\boldsymbol{E}\imath x\alpha \rightarrow \forall x(\alpha \rightarrow x = \imath x\alpha)$
- $\boldsymbol{E}\imath x\alpha \rightarrow \alpha(\imath x\alpha/x)$ (**restricted characterization principle**)
- $\boldsymbol{E}\imath x\alpha \rightarrow \exists x\alpha$
- $\boldsymbol{E}\imath x\alpha \rightarrow \forall x\forall y[(\alpha \wedge \alpha(y/x)) \rightarrow x = y]$
- $\{\exists x\alpha \wedge \forall x\forall y[(\alpha \wedge \alpha(y/x)) \rightarrow x = y]\} \rightarrow \boldsymbol{E}\imath x\alpha$
- $\boldsymbol{E}\imath x\alpha \leftrightarrow \exists y\forall x(\alpha \leftrightarrow x = y)$ (**Russell's equivalence I**)
- $\exists y[\forall x(\alpha \leftrightarrow x = y) \wedge \beta] \rightarrow \beta(\imath x\alpha/y)$
- $\boldsymbol{E}\imath x\alpha \rightarrow \{\beta(\imath x\alpha/y) \leftrightarrow \exists y[\forall x(\alpha \leftrightarrow x = y) \wedge \beta]\}$ (**Russell's equivalence II**)
- $\boldsymbol{E}\imath x\alpha \rightarrow [\beta(\imath x\alpha/x) \leftrightarrow \forall x(\alpha \rightarrow \beta)]$
- $(\boldsymbol{E}\imath x\alpha \vee \boldsymbol{E}\imath x\beta) \rightarrow [\forall x(\alpha \leftrightarrow \beta) \rightarrow \imath x\alpha = \imath x\beta]$
- $\forall y(\imath x(x = y) = y)$ (**restricted cancellation principle**)
- $\neg\boldsymbol{E}\imath x(x \neq x)$

($\alpha(t/x)$ is the result of replacing all free occurrences of x in α by t)

Free theories of definite descriptions

- **Minimal free theory of DDs** (mFD): positive free logic + LL
- **Maximal free theory of DDs** (MFD/FD2): mFD +

Identity of nonexistents

(FD2) $$(\neg \boldsymbol{E} t_1 \wedge \neg \boldsymbol{E} t_2) \to t_1 = t_2$$

(Scott [29]; Leblanc & Thomason [19])

- **Free Russellian theory of DDs** (RFD): **negative free logic** + LL +

$$[x\colon \alpha]t \leftrightarrow (\boldsymbol{E} t \wedge \alpha(t/x))$$

(Scales [28]; Lambert [16])

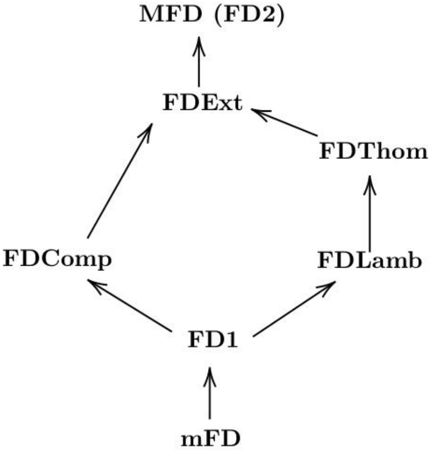
- [:] is a **variable binding predicate operator** that allows expressing scope distinctions in RFD.

Free theories of definite descriptions

van Fraassen's spectrum of free theories of DDs (Lambert [17])

- $\imath x(x = t) = t$ (FD1)
  - $\alpha(\imath x\alpha/x)$ holds when α is the formula $x = t$.
- $\neg \boldsymbol{E}\imath x\alpha \rightarrow \imath x\alpha = \imath x(x \neq x)$ (FDScott)
  - Every improper DD is identical to $\imath x(x \neq x)$.
- $(\alpha(t/x) \wedge \neg\exists x\alpha) \rightarrow \alpha(\imath x\alpha/x)$ (FDLamb)
  - For $\alpha(\imath x\alpha/x)$ to hold it suffices that α holds of at least one nonexistent and that it holds of no existent whatsoever.
- $(\alpha(t/x) \wedge \neg\boldsymbol{E}t) \rightarrow \alpha(\imath x\alpha/x)$ (FDThom)
  - For $\alpha(\imath x\alpha/x)$ to hold it suffices that α holds of at least one nonexistent, irrespective of whether some existent also satisfies α.
- $\forall x(\alpha \leftrightarrow \beta) \rightarrow \imath x\alpha = \imath x\beta$ (FDExt)
  - If α and β are co-extensional, then $\imath x\alpha$ is identical with $\imath x\beta$.
- $\Pi x(\alpha \leftrightarrow \beta) \rightarrow \imath x\alpha = \imath x\beta$ (FDComp)
  - If α and β are co-comprehensive, then $\imath x\alpha$ is identical with $\imath x\beta$.

Free theories of definite descriptions



Definite description paradoxes I

Characterization principle

(CP) $$\alpha(\imath x\alpha/x)$$

- Since $\boldsymbol{E}\imath x\alpha \rightarrow \alpha(\imath x\alpha/x)$ is a theorem of mFD, CP holds for every proper DD. However, mFD does not lay down any conditions under which CP is supposed to hold when $\imath x\alpha$ is improper.
- Some theories in the spectrum explicitly extend mFD in this regard, e.g.:
  - $\imath x(x = t) = t$ (FD1)
  - $(\alpha(t/x) \wedge \neg\exists x\alpha) \rightarrow \alpha(\imath x\alpha/x)$ (FDLamb)
  - $(\alpha(t/x) \wedge \neg\boldsymbol{E}t) \rightarrow \alpha(\imath x\alpha/x)$ (FDThom)
- As Lambert [15] has shown, there is a very good reason why CP cannot be taken to hold in its full generality: it is inconsistent!

Definite description paradoxes I

- **Paradox I:** Let α be the formula $Fx \wedge \neg Fx$. It follows from CP that:

$$F\imath x(Fx \wedge \neg Fx) \wedge \neg F\imath x(Fx \wedge \neg Fx)$$

- **Paradox II ("Curry's Paradox"):** Consider the following formula:

$$x = x \rightarrow \beta$$

It follows from CP that:

$$\imath x(x = x \rightarrow \beta) = \imath x(x = x \rightarrow \bot) \rightarrow \beta$$

If $t = t$ is a valid schema of the underlying logic, one can infer β by *modus ponens*, for any β.

- In some free logics $t = t$ is not a valid schema. This does not prevent the derivation of the above paradox, though, for it suffices to replace $x = x$ by any tautological formula α in which x is free.

Definite description paradoxes I

- CP cannot be assumed to hold in an **explosive logic**, i.e., a logic in which the following schema is valid:

Principle of Explosion

(EXP) $$\alpha, \neg\alpha \vDash \beta$$

For otherwise the conclusion of Paradox I would lead to triviality.

- Paradox II shows that even if the underlying logic is non-explosive (i.e., is paraconsistent), CP leads to triviality provided that the logic has an implication connective $\rightarrow$ w.r.t. which *modus ponens* is valid.
- What if the underlying logic (i) is paraconsistent and (ii) lacks an implication connective w.r.t. which *modus ponens* is valid?
  - Free version of *FDE*
  - Free version of *LP*

The logic *FFDE*

- In [3] Abilio Rodrigues and I propose universally free versions with variable domains of the logics *FDE* and *N*4, called respectively *FFDE* and *FN*4.
- *FDE* was investigated in a series of papers by Belnap and Dunn in the 1970s [6, 7, 14], while *N*4 is the sentential fragment of the logic N^{-}, proposed by Almukdad and Nelson in [1]. *N*4 is a paraconsistent version of Nelson's logic *N*3 [21].
- Both systems, *FFDE* and *FN*4, are interpreted as **information-based logics** in [3], i.e., logics that are supposed to be used for processing information in the sense of allowing one (viz., a person or a computer) to draw sensible conclusions from the information stored in a given database.
- This interpretation is in line with the intepretation of *FDE* advanced by Belnap in [6].

The logic *FFDE*

FFDE and *FN*4 are better suited to represent information states that:

- May be either contradictory or incomplete;
  - *FFDE* and *FN*4 are both paraconsistent and paracomplete.
- Evolve over time, in the sense that they can be supplemented with new pieces of information as time goes by;
  - Both systems are endowed with Kripke semantics that resemble that of intuitionistic logic.
- Are such that the new information may concern previously unacknowledged individuals.
  - The stages of a model have variable, but $\leq$-cumulative, domains;
  - The domain of a stage may be empty;
  - Individual constants may be empty;

Both *FFDE* and *FN*4 are **negative** logics in the sense that every atomic formula $Pc_1 \ldots c_n$ (and negation therefore) in which an empty term occurs receives a non-designated value, which is supposed to represent the lack of information about $Pc_1 \ldots c_n$.

The logic *FFDE*

- Given a stage w of a model $\mathcal{M}$, which is supposed to represent a stage of the development of a certain database, a formula α and its negation $\neg\alpha$ may receive either the value 1 or the value 0 at w. The values of α and $\neg\alpha$ are independent of each other.
- The value 1 represents the presence of information, either positive or negative, while the value 0 represents the lack thereof.
- There are four possibilities:
  - $v(\alpha, w) = 1$ and $v(\neg\alpha, w) = 0$
    - ★ w contains the information that α is true, but it doesn't contain the information that α is false.
  - $v(\alpha, w) = 0$ and $v(\neg\alpha, w) = 1$
    - ★ w contains the information that α is false, but it doesn't contain the information that α is true.
  - $v(\alpha, w) = 0$ and $v(\neg\alpha, w) = 0$
    - ★ w contains no information whatsoever concerning α. (lack of information)
  - $v(\alpha, w) = 1$ and $v(\neg\alpha, w) = 1$
    - ★ w contains the information that α is true and it also contains the information that it is false. (contradictory information)

The logic *FFDE*

- The values of an atomic sentence $Pc_1 \ldots c_n$ and its negation $\neg Pc_1 \ldots c_n$ at w are determined by P's **extension** P^w_+ and its **anti-extension** P^w_- at w.
  - P^w_+ is the set of n-tuples of individuals (in $d(w)$) that satisfy P at w;
  - P^w_- is the set of n-tuples of individuals (in $d(w)$) that do not satisfy P at w;
- P^w_+ and P^w_- are required to be either or exhaustive w.r.t. the domain of w, i.e.:
  - $P^w_+ \cap P^w_-$ may be nonempty;
  - $P^w_+ \cup P^w_-$ may be a proper subset of the domain of w.
- The domains of stages and the extensions and anti-extensions of predicates are preserved across $\leq$-related stages:
  - If $w \leq w'$, then $d(w) \subseteq d(w')$;
  - If $w \leq w'$, then $P^w_+ \subseteq P^{w'}_+$ and $P^w_- \subseteq P^{w'}_-$.
- The values of the remaining sentences are determined by the values of their subformulas and negations thereof. For example:
  - $v(\alpha \wedge \beta, w) = 1$ iff $v(\alpha, w) = 1$ and $v(\beta, w) = 1$;
  - $v(\neg(\alpha \vee \beta), w) = 1$ iff $v(\neg\alpha, w) = 1$ and $v(\neg\beta, w) = 1$.

The logic *FFDE*

- The interpretation function $\mathcal{I}$ of a model $\mathcal{M}$ may be either defined or undefined at c and w. In the former case, $\mathcal{I}(c, w)$ is an element of $d(w)$.
- If c is empty at w, i.e., if $\mathcal{I}(c, w)$ is undefined, then every atomic formula and negation thereof in which c occurs receives the value 0 at w. This is supposed to represent the idea that there can be no information, either positive or negative, involving empty names.
- Individual constants are **rigid designators**, they are interpreted as the same individual across $\leq$-related stages:
  - If $w \leq w'$ and $\mathcal{I}(c, w)$ is defined, then $\mathcal{I}(c, w') = \mathcal{I}(c, w)$.
- The language of *FFDE* includes a **definedness predicate *E*** that expresses whether $\mathcal{I}(c, w)$ is defined:
  - $v(\boldsymbol{E}c, w) = 1$ iff $\mathcal{I}(c, w)$ is defined;
  - $v(\neg \boldsymbol{E}c, w) = 1$ iff $\mathcal{I}(c, w)$ is undefined;
- Hence, ***E*** behaves classically in *FFDE*, that is, the following two possibilites are ruled out:
  - $v(\boldsymbol{E}c, w) = 0$ and $v(\boldsymbol{E}c, w) = 0$;
  - $v(\boldsymbol{E}c, w) = 1$ and $v(\boldsymbol{E}c, w) = 1$.

The logic *FFDE*

- The language of *FFDE* also includes an identity predicate $=$. The extension $i_+(w)$ of $=$ at w is a **congruence relation** on w, that is, an equivalence relation such that:
  - For every n-ary predicate letter P and every $a_1, \ldots, a_n, b_1, \ldots, b_n \in d(w)$, if $\langle a_i, b_i \rangle \in i_+(w)$, for every $1 \leq i \leq n$, then:
    - ★ $\langle a_1, \ldots, a_n \rangle \in P_+^w$ iff $\langle b_1, \ldots, b_n \rangle \in P_+^w$;
    - ★ $\langle a_1, \ldots, a_n \rangle \in P_-^w$ iff $\langle b_1, \ldots, b_n \rangle \in P_-^w$;
- $=$'s anti-extension $i_-(w)$ at w is any binary relation on $d(w)$ that satisfies the following condition:
  - If there is an n-ary predicate letter P such that $\langle \ldots a \ldots \rangle \in P_+^w$ and $\langle \ldots b \ldots \rangle \in P_-^w$, then $\langle a, b \rangle \in i_-(w)$ and $\langle b, a \rangle \in i_-(w)$.
- The conditions placed on $i_+(w)$ are supposed to ensure the validity of the usual rules of identity, viz., $= I$ and $= E$ (**indiscernibility of identicals**).
- The conditions placed on $i_-(w)$ are supposed to ensure the validity of the following rule:

$$Pc_1, \neg Pc_2 \vDash c_1 \neq c_2$$

The logic *FFDE* (skip)

Model

Let $\mathcal{L} = \langle \mathcal{C}, \mathcal{P} \rangle$ be a first-order language. A (**Kripke** *FFDE*-)**model** $\mathcal{K}$ for $\mathcal{L}$ is a structure $\langle W, \leq, d, I, i_+, i_- \rangle$ such that: (i) W is a non-empty set of **stages**; (ii) $\leq$ (**accessibility relation**) is a pre-order on W; (iii) d is a function on W such that $d(w)$ is a set (the **domain** of w) satisfying the following condition: if $w \leq w'$, then $d(w) \subseteq d(w')$; (iv) I (**interpretation function**) is a partial function on $(\mathcal{C} \cup \mathcal{P}) \times W$ such that:

- For every individual constant $c \in \mathcal{C}$, if $I(c, w)$ is defined, then $I(c, w) \in d(w)$;
- For every individual constant $c \in \mathcal{C}$, if $I(c, w)$ is defined, then $I(c, w) = I(c, w')$, for every $w' \geq w$;
- For every m-ary predicate letter $P \in \mathcal{P}$, $I(P, w)$ is the pair $\langle P^w_+, P^w_- \rangle$ such that $P^w_+, P^w_- \subseteq d(w)^m$ and if $w \leq w'$, then $P^w_+ \subseteq P^{w'}_+$ and $P^w_- \subseteq P^{w'}_-$.

The logic *FFDE* (skip)

Model

(v) i_+ and i_- are functions on W such that $i_+(w)$ and $i_-(w)$ are binary relations on $d(w)$ that satisfy the following conditions:

(a) $i_+(w)$ is a congruence relation on $d(w)$. That is, $i_+(w)$ is an equivalence relation such that for every $P \in \mathcal{P}$ and $a_1, \ldots, a_m, a'_1, \ldots, a'_m \in d(w)$, if $\langle a_j, a'_j \rangle \in i_+(w)$, for every $1 \leq j \leq m$, then $\langle a_1, \ldots, a_m \rangle \in P^w_+$ iff $\langle a'_1, \ldots, a'_m \rangle \in P^w_+$, and $\langle a_1, \ldots, a_m \rangle \in P^w_-$ iff $\langle a'_1, \ldots, a'_m \rangle \in P^w_-$;

(b) For every $a, a' \in d(w)$, if there is an m-ary predicate $P \in \mathcal{P}^=$ and an m-tuple $\langle \ldots a \ldots \rangle$ of elements of $d(w)$ such that $\langle \ldots a \ldots \rangle \in P^w_+$ and $\langle \ldots a' \ldots \rangle \in P^w_-$, then $\langle a, a' \rangle \in i_-(w)$ and $\langle a', a \rangle \in i_-(w)$.

(c) For every $w, w' \in W$, if $w \leq w'$, then $i_+(w) \subseteq i_+(w')$ and $i_-(w) \subseteq i_-(w')$.

The logic *FFDE* (skip)

Valuation

Let $\mathcal{L}$ be a first-order language and let $\mathcal{K}$ be a Kripke model for $\mathcal{L}$. The **valuation function induced by** $\mathcal{K}$ is the mapping $v\colon Sent(\mathcal{L}_{\mathcal{K}}) \times W \longrightarrow \{0,1\}$ satisfying the following conditions:

($v1$) $v(\boldsymbol{E}c, w) = 1$ iff $\widehat{\mathcal{I}}(c, w)$ is defined.

($v2$) $v(\neg\boldsymbol{E}c, w) = 1$ iff $\widehat{\mathcal{I}}(c, w)$ is undefined.

($v3$) $v(c_1 = c_2, w) = 1$ iff $\widehat{\mathcal{I}}(c_1, w)$ and $\widehat{\mathcal{I}}(c_2, w)$ are defined and $\langle\widehat{\mathcal{I}}(c_1, w), \widehat{\mathcal{I}}(c_1, w)\rangle \in i_+(w)$;

($v4$) $v(c_1 \neq c_2, w) = 1$ iff $\widehat{\mathcal{I}}(c_1, w)$ and $\widehat{\mathcal{I}}(c_2, w)$ are defined and $\langle\widehat{\mathcal{I}}(c_1, w), \widehat{\mathcal{I}}(c_1, w)\rangle \in i_-(w)$;

($v5$) For every m-ary predicate letter $P \in \mathcal{P}$, $v(Pc_1 \dots c_m, w) = 1$ iff $\widehat{\mathcal{I}}(c_i, w)$ is defined, for every $1 \leq i \leq m$, and $\langle\widehat{\mathcal{I}}(c_1, w), \dots, \widehat{\mathcal{I}}(c_m, w)\rangle \in P^w_+$;

($v6$) For every m-ary predicate letter $P \in \mathcal{P}$, $v(\neg Pc_1 \dots c_m, w) = 1$ if $\widehat{\mathcal{I}}(c_i, w)$ is defined, for every $1 \leq i \leq m$, and $\langle\widehat{\mathcal{I}}(c_1, w), \dots, \widehat{\mathcal{I}}(c_m, w)\rangle \in P^w_-$;

The logic *FFDE* (skip)

Valuation

(v7) $v(A \wedge B, w) = 1$ iff $v(A, w) = 1$ and $v(B, w) = 1$;

(v8) $v(A \vee B, w) = 1$ iff $v(A, w) = 1$ or $v(B, w) = 1$;

(v9) $v(\neg(A \wedge B), w) = 1$ iff $v(\neg A, w) = 1$ or $v(\neg B, w) = 1$;

(v10) $v(\neg(A \vee B), w) = 1$ iff $v(\neg A, w) = 1$ and $v(\neg B, w) = 1$;

(v11) $v(\neg\neg A, w) = 1$ iff $v(A, w) = 1$;

(v12) $v(\forall x A, w) = 1$ iff for every $w' \geq w$, $v(A(\overline{a}/x), w') = 1$, for every $a \in d(w')$;

(v13) $v(\exists x A, w) = 1$ iff $v(A(\overline{a}/x), w) = 1$, for some $a \in d(w)$;

(v14) $v(\neg\forall x A, w) = 1$ iff $v(\neg A(\overline{a}/x), w) = 1$, for some $a \in d(w)$;

(v15) $v(\neg\exists x A, w) = 1$ iff for every $w' \geq w$, $v(\neg A(\overline{a}/x), w') = 1$, for every $a \in d(w')$.

The logic *FFDE*

Positive sentential rules

$$\frac{\alpha \qquad \beta}{\alpha \wedge \beta} \wedge I \qquad \frac{\alpha \wedge \beta}{\alpha} \wedge E \ \frac{\alpha \wedge \beta}{\beta}$$

$$\frac{\alpha}{\alpha \vee \beta} \vee I \ \frac{\beta}{\alpha \vee \beta} \qquad \frac{\alpha \vee \beta \qquad \begin{array}{c}[\alpha]^i \\ \vdots \\ \gamma\end{array} \qquad \begin{array}{c}[\beta]^i \\ \vdots \\ \gamma\end{array}}{\gamma} \vee E^i$$

The logic *FFDE*

Negative sentential rules

$$\frac{\neg\alpha}{\neg(\alpha\wedge\beta)}\ \neg\wedge I\ \frac{\neg\beta}{\neg(\alpha\wedge\beta)} \qquad \frac{\neg(\alpha\wedge\beta)\qquad \begin{array}{c}[\neg\alpha]^{i}\\ \vdots\\ \gamma\end{array}\qquad \begin{array}{c}[\neg\beta]^{i}\\ \vdots\\ \gamma\end{array}}{\gamma}\ \neg\wedge E^{i}$$

$$\frac{\neg\alpha\qquad\neg\beta}{\neg(\alpha\vee\beta)}\ \neg\vee I \qquad \frac{\neg(\alpha\vee\beta)}{\neg\alpha}\ \neg\vee E\ \frac{\neg(\alpha\vee\beta)}{\neg\beta}$$

$$\frac{\alpha}{\neg\neg\alpha}\ DN\ \frac{\neg\neg\alpha}{\alpha}$$

The logic *FFDE*

Positive quantifier rules

$$\dfrac{\begin{matrix}[\boldsymbol{E}c]^i \\ \vdots \\ \alpha(c/x)\end{matrix}}{\forall x\alpha}\ \forall I^i \qquad \dfrac{\forall x\alpha \qquad \boldsymbol{E}c}{\alpha(c/x)}\ \forall E$$

$$\dfrac{\alpha(c/x) \qquad \boldsymbol{E}c}{\exists x\alpha}\ \exists I \qquad \dfrac{\exists x\alpha \qquad \begin{matrix}[\alpha(c/x), \boldsymbol{E}c]^i \\ \vdots \\ \beta\end{matrix}}{\beta}\ \exists E^i$$

The logic *FFDE*

Negative quantifier rules

$$\frac{\neg\alpha(c/x) \qquad \boldsymbol{E}c}{\neg\forall x\alpha}\ \neg\forall I \qquad\qquad \frac{\neg\forall x\alpha \qquad \begin{array}{c}[\neg\alpha(c/x), \boldsymbol{E}c]^i \\ \vdots \\ \beta\end{array}}{\beta}\ \neg\forall E^i$$

$$\frac{\begin{array}{c}[\boldsymbol{E}c]^i \\ \vdots \\ \neg\alpha(c/x)\end{array}}{\neg\exists x\alpha}\ \neg\exists I^i \qquad\qquad \frac{\neg\exists x\alpha \qquad \boldsymbol{E}c}{\neg\alpha(c/x)}\ \neg\exists E$$

The logic *FFDE*

Identity rules

$$\frac{\boldsymbol{E}c}{c = c} = I$$

$$\frac{\alpha(c_1/x) \qquad c_1 = c_2}{\alpha(c_2/x)} = E$$

(α is an atomic formula or the negation of an atomic formula)

$$\frac{\alpha(c_1/x) \qquad \neg\alpha(c_2/x)}{c_1 \neq c_2} \neq I$$

(α is an atomic formula)

The logic *FFDE*

$\boldsymbol{E}$-rules

$$\frac{\boldsymbol{E}c \qquad \neg\boldsymbol{E}c}{\beta}\ PEX_{\boldsymbol{E}} \qquad \frac{}{\boldsymbol{E}c \vee \neg\boldsymbol{E}c}\ PEM_{\boldsymbol{E}}$$

$$\frac{Pc_1 \ldots c_m}{\boldsymbol{E}c_i}\ \boldsymbol{E}I\ \frac{\neg Pc_1 \ldots c_m}{\boldsymbol{E}c_i}$$

- Rules $PEX_{\boldsymbol{E}}$ and $PEM_{\boldsymbol{E}}$ correspond to the fact that $\boldsymbol{E}$ behaves classically in *FFDE*.
- The left version of rule $\boldsymbol{E}I$ corresponds to the fact that *FFDE* is a negative free logc.
- The contrapositive of the left version of rule $\boldsymbol{E}I$ does not hold:

$$\frac{\neg\boldsymbol{E}c}{\neg Pc_1 \ldots c_m}$$

The logic *FFDE* (skip)

Generalized $=E$ and $\neq I$

- If α has at most x free, then

$$\alpha(c_1/x), c_1 = c_2 \vDash \alpha(c_2/x)$$

- If x is free in every subformula of α, then

$$\alpha(c_1/x), \neg\alpha(c_2/x) \vDash c_1 \neq c_2$$

Counter-example

Let $\mathcal{K} = \langle W, \leq, d, I, i_+, i_- \rangle$ be an *FFDE*-model such that $W = \{w\}$, $d(w) = \{1, 2\}$, $i_+(w) = \{\langle 1, 1\rangle, \langle 2, 2\rangle\}$, and $i_-(w) = \{\langle 2, 2\rangle\}$. Suppose that $I(c_1, w) = 1$ and $I(c_2, w) = 2$. Suppose further that $Q^w_+ = \{2\}$ and $Q^w_- = \{2\}$. Thus, $\mathcal{K}, w \vDash Qc_2$ and $\mathcal{K}, w \vDash \neg Qc_2$. Letting $P^w_+ = \{1\}$ and $P^+_- = \varnothing$, it then follows that $\mathcal{K}, w \vDash Pc_1 \wedge Qc_2$ and $\mathcal{K}, w \vDash \neg(Pc_1 \wedge Qc_2)$. Therefore, even though both sentences hold in $\mathcal{K}$ and w, $c_1 \neq c_2$ does not.

Definite description paradoxes II

- Paradox I shows that CP cannot be assumed to hold in an explosive logic.
- Paradox II shows that even if the underlying logic is non-explosive (i.e., is paraconsistent), CP leads to triviality provided that the logic has an implication connective $\rightarrow$ w.r.t. which *modus ponens* is valid.
- What if the underlying logic (i) is paraconsistent and (ii) lacks an implication connective w.r.t. which *modus ponens* is valid?
- Let *PFFDE*, which is the positive version of *FFDE*, be the logic that results from dropping both versions of rule $\boldsymbol{E}I$.
- Is the theory that results from adding CP to *PFFDE* non-trivial?
- Unless CP is subject to certain restrictions, the answer is "No".

Definite description paradoxes II

- First, since $\boldsymbol{E}$ behaves classically, the following instance of CP leads immediatly to triviality (by $PEM_{\boldsymbol{E}}$):

$$\boldsymbol{E}\imath x(\boldsymbol{E}x \wedge \neg \boldsymbol{E}x) \wedge \neg \boldsymbol{E}\imath x(\boldsymbol{E}x \wedge \neg \boldsymbol{E}x)$$

- Second, $PFFDE$ + CP is trivial even if $\boldsymbol{E}$ is required not to occur in α.

Let β be any sentence and consider the following formula:

$$x = x \wedge \beta$$

By CP, it follows that:

(*) $$\imath x(x = x \wedge \beta) = \imath x(x = x \wedge \beta) \wedge \beta$$

By applying $\wedge E$ to $(*)$, one can then infer β.

Definite description paradoxes II

- One might hope that requiring x to be free in every subformula of α in CP would avoid the paradox above.
- This is true, but this move won't be capable of avoiding the following undesirable result:

Consider the following formula:

$$\forall y(y = x)$$

By CP, it follows that:

$$\forall y(y = \imath x \forall y(y = x))$$

In then follows that if there exists at least one object, then there exists exactly one object:

$$\exists y \boldsymbol{E} y \vdash \exists x \forall y(y = x)$$

Definite description paradoxes II

- The negative results above carry over to every extension of *PFFDE*, such as the positive free versions of *LP* (the **logic of paradox** [25]) and *N*4.
- Since *FDE* is a rather weak logic, there is little hope of coming up with a non-trivial free theory of DDs one of whose axioms is CP seems to be a hopeless task.
- One can still formulate more modest theories of DDs that are based on *FDE* and some of its extensions.
- These theories include inference rules that correspond to some of the axioms and theorems of the theories in the spectrum of free theories of DDs.

Free theories of DDs based on *FDE*

Minimal theory of DDs based on *FDE*

Let mFD_F be the system that results from adding the following rules to *PFFDE*:

$$\frac{\boldsymbol{E}\imath x\alpha}{\exists x\alpha}\ PR1 \qquad \frac{\boldsymbol{E}\imath x\alpha \qquad \boldsymbol{E}t \qquad \alpha(t/x)}{t = \imath x\alpha}\ PR2$$

$$\frac{\exists x\alpha \qquad \begin{matrix}[\boldsymbol{E}x, \alpha, \boldsymbol{E}y, \alpha(y/x)]^i \\ \vdots \\ x = y\end{matrix}}{\boldsymbol{E}\imath x\alpha}\ PR3^i$$

- *PR*1, *PR*2, and *PR*3 correspond respectively to the following theorems of mFD:

$$\boldsymbol{E}\imath x\alpha \rightarrow \exists x\alpha$$

$$\boldsymbol{E}\imath x\alpha \rightarrow \forall x(\alpha \rightarrow x = \imath x\alpha)$$

$$\{\exists x\alpha \wedge \forall x\forall y[(\alpha \wedge \alpha(y/x)) \rightarrow x = y]\} \rightarrow \boldsymbol{E}\imath x\alpha$$

Free theories of DDs based on *FDE*

A few consequences:

- $\boldsymbol{E}\imath x\alpha \vdash \imath x\alpha = \imath x\alpha$
- $\boldsymbol{E}\imath x\alpha \vdash \alpha(\imath x\alpha/x)$ **(restricted characterization principle)**
- $\vdash \forall y(\imath x(x = y) = y)$ **(restricted cancellation principle)**
- The following rule is derivable in mFD_F:

$$\frac{\boldsymbol{E}\imath x\alpha \qquad \boldsymbol{E}t_1 \qquad \alpha(t_1/x) \qquad \boldsymbol{E}t_2 \qquad \alpha(t_2/x)}{t_1 = t_2}\ DR1$$

- $DR1$ corresponds to the following theorem of mFD:

$$\boldsymbol{E}\imath x\alpha \to \forall x\forall y[(\alpha \wedge \alpha(y/x)) \to x = y]$$

Free theories of DDs based on *FDE*

- The following rules are derivable in mFD_F:

$$\dfrac{\boldsymbol{E}\imath x\alpha \qquad \beta(\imath x\alpha/x) \qquad \boldsymbol{E}t \qquad \alpha(t/x)}{\beta(t/x)}\ DR2$$

$$\dfrac{\boldsymbol{E}\imath x\alpha \qquad \begin{array}{c}[\boldsymbol{E}x, \alpha] \\ \vdots \\ \beta\end{array}}{\beta(\imath x\alpha/x)}\ DR3$$

- *DR*2 and *DR*3 correspond respectively to the left-to-right and to the right-to-left directions of the consequent of the following theorem of mFD:

$$\boldsymbol{E}\imath x\alpha \rightarrow [\beta(\imath x\alpha/x) \leftrightarrow \forall x(\alpha \rightarrow \beta)]$$

Free theories of DDs based on *FDE*

- The following rule can be derived in mFD_F:

$$\dfrac{\exists x\alpha \qquad \begin{matrix}[\boldsymbol{E}x, \alpha, \boldsymbol{E}y, \alpha(y/x)]^i \\ \vdots \\ x = y\end{matrix} \qquad \begin{matrix}[\boldsymbol{E}x, \alpha]^i \\ \vdots \\ \beta\end{matrix}}{\beta(\imath x\alpha/x)}\ DR4^i$$

- *DR*4 corresponds to the following theorem of mFD:

$$\{\exists x\alpha \wedge \forall x \forall y[(\alpha \wedge \alpha(y/x)) \rightarrow x = y] \wedge \forall x(\alpha \rightarrow \beta)\} \rightarrow \beta(\imath x\alpha/x)$$

which, in turn, is equivalent to the right-to-left direction of the unconditionalized version of Russell's equivalence II:

$$\exists y[\forall x(\alpha \leftrightarrow x = y) \wedge \beta] \rightarrow \beta(\imath x\alpha/y)$$

Free theories of DDs based on *FDE*

- The following rule is derivable in mFD_F:

$$\dfrac{\boldsymbol{E}\imath x\alpha \vee \boldsymbol{E}\imath x\beta \qquad \begin{matrix}[\boldsymbol{E}x, \alpha]^i \\ \vdots \\ \beta\end{matrix} \qquad \begin{matrix}[\boldsymbol{E}x, \beta]^i \\ \vdots \\ \alpha\end{matrix}}{\imath x\alpha = \imath x\beta}\ DR5^i$$

- *DR*5 corresponds to the following theorem of mFD:

$$(\boldsymbol{E}\imath x\alpha \vee \boldsymbol{E}\imath x\beta) \rightarrow [\forall x(\alpha \leftrightarrow \beta) \rightarrow \imath x\alpha = \imath x\beta]$$

Free theories of DDs based on *FDE*

- One can obtain a Russellian free theory of DDs by adding *PR*1-*PR*3 and the following rules to *FFDE*:

$$\frac{[x\colon\alpha]t}{\boldsymbol{E}t} \qquad \frac{[x\colon\alpha]t}{\alpha(t/x)} \qquad \frac{\boldsymbol{E}t \qquad \alpha(t/x)}{[x\colon\alpha]t}$$

- Also, by adding suitable rules to mFD_F one can obtain theories that resemble those in van Fraassen's spectrum.
- For example, one can obtain versions of FDLamb and FDThom that are based on *PFFDE* by adding the following rules to mFD_F, respectively:

$$\frac{\alpha(t/x) \qquad \forall x\neg\boldsymbol{E}x}{\alpha(\imath x\alpha/x)}\ \text{FDLamb} \qquad \frac{\alpha(t/x) \qquad \neg\boldsymbol{E}t}{\alpha(\imath x\alpha/x)}\ \text{FDThom}$$

The road ahead

- Formulate a formal semantics w.r.t. which mFD_F is both sound and complete.
- Formulate negative natural deduction rules for $\imath$.

$$\frac{\neg\exists x\alpha}{\neg \boldsymbol{E}\imath x\alpha} \qquad \frac{\exists x\exists y(\alpha\wedge\alpha(y/x)\wedge x\neq y)}{\neg \boldsymbol{E}\imath x\alpha}$$

- The above rules won't work, for while the occurrences of $\neg$ in the premises are paraconsistent and paracomplete, its occurrences in the conclusions behave classically.
- Specifically, $\neg\exists xFx$ holds if every element of the domain of a stage belongs to the anti-extension of F. However, this does not prevent the extension of F from being a singleton.
- Similarly, $\exists x\exists y(Fx\wedge Fy\wedge x\neq y)$ can hold even if the extension of F is a singleton $\{a\}$, for a may be such that $\langle a,a\rangle$ belong both to the extension and to the anti-extension of $=$.

The road ahead

- Investigate the properties of the *FDE*-versions of the therioes in van Fraassen's spectrum, as well as the relationships among them.
- Formulate free theories of DDs that are based on non-classical extensions of *FDE*:
    - *LP* [25] and *N4* [1]
    - **Logics of formal inconsistency and undeterminedness** (*LFIUs*)
        - ★ *LFI*1 [9, 11, 13] and *BS*4 [23, 24]
    - **Logics of evidence and truth** (*LET*s)
        - ★ LET_F, LET_J, LET_K [4, 10, 27]
        - ★ LET_F^+ and LET_K^+ [12]
- Investigate if and how formal theories of DDs based on *LFIU*s can be leveraged to yield a philosophically well-motivated formal rendering of a **Meinongian theory of objects.**

Thanks!

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