Constructive Proof of the Craig Interpolation Theorem for Russellian Logic of Definite Descriptions

Andrzej Indrzejczak

Department of Logic, University of Lodz

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3. Sequent calculus equivalent to KMM.
4. Interpolation theorem for RDD.

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Definite descriptions represented by means of iota-operator (first appeared in Peano):

$\imath x\varphi(x)$ – the (unique) object x being φ

$\psi(\imath x\varphi(x))$ – The only x being φ is ψ.

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How to avoid problems?

Show that definite descriptions are not genuine names.

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- rejection of intuitively valid formulae;
- running into contradiction.

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In PM special scope operators attached.

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which implies:

$\psi(\imath x\varphi(x)) \to E\imath x\varphi(x)$

On the other hand:

$\imath x\varphi(x) = \imath x\varphi(x)$ does not hold for improper descriptions.

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$$\Downarrow$$

$$A(\imath x(Bx \land \neg Bx)) \lor \neg A(\imath x(Bx \land \neg Bx)) \;:=$$
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leads to

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2. $\forall x(Ax \wedge \neg Ax \leftrightarrow x = \imath x(Ax \wedge \neg Ax))$ 1
3. $A(\imath x(Ax \wedge \neg Ax)) \wedge \neg A(\imath x(Ax \wedge \neg Ax)) \leftrightarrow \imath x(Ax \wedge \neg Ax) = \imath x(Ax \wedge \neg Ax))$ 2
4. $A(\imath x(Ax \wedge \neg Ax)) \wedge \neg A(\imath x(Ax \wedge \neg Ax))$ 3, 1

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Two solutions:

- Kalish, Montague and Mar [1980] (also Burge [1974], Feferman [1995]) $\Longrightarrow$ Indrzejczak [RSL 2022]
- use lambda-operator $\Longrightarrow$ [Indrzejczak 2020, Indrzejczak and Zawidzki 2022, Indrzejczak and Kürbis 2023]

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4. taking object language counterpart of Russellian definition as an axiom characterising descriptions (we will call it RA):

RA $\psi[x/\imath y\varphi] \leftrightarrow \exists x(\forall y(\varphi \leftrightarrow y = x) \wedge \psi)$ with ψ atomic.

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Note that if equalities are treated as atomic then RA implies:

$(DP\text{-}1)$ $R^n t_1 ... t_n \vdash \exists x x = t_1 \wedge ... \wedge \exists x x = t_n$

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moreover, due to $(DP\text{-}1)$ RA is equivalent to:
LA $\forall y(y = \imath x\varphi(x) \leftrightarrow \forall x(\varphi(x) \leftrightarrow x = y))$

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(ID)	$\varnothing \vdash b = b$
(LL)	$t_1 = t_2, \varphi[x/t_1] \vdash \varphi[x/t_2]$
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How to change it into well-behaved SC?

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the subformula property lost.

Sequents versus rules

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For any sequent $\Gamma \Rightarrow \Delta$ with $\Gamma = \{\varphi_1, ..., \varphi_k\}$ and $\Delta = \{\psi_1, ..., \psi_n\}, k \geq 0, n \geq 0$ there is $2^{k+n} - 1$ equivalent rules captured by the general schema:

$$\frac{\Pi_1, \Rightarrow \Sigma_1, \varphi_1, \ \ ..., \ \ \Pi_i \Rightarrow \Sigma_i, \varphi_i \qquad \psi_1, \Pi_{i+1} \Rightarrow \Sigma_{i+1}, \ \ ..., \ \ \psi_j, \Pi_{i+j} \Rightarrow \Sigma_{i+j}}{\Gamma^{-i}, \Pi_1, ..., \Pi_i, \Pi_{i+1}, ..., \Pi_{i+j} \Rightarrow \Sigma_1, ..., \Sigma_i, \Sigma_{i+1}, ..., \Sigma_{i+j}\Delta^{-j}}$$

where $\Gamma^{-i} = \Gamma - \{\varphi_1, ..., \varphi_i\}$ and $\Delta^{-j} = \Delta - \{\psi_1, ..., \psi_j\}$ for $0 \leq i \leq k, \ 0 \leq j \leq n$.

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Provide cut-free SC with rules for DD possibly close to standard ones.

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Each may be changed into introduction rule by RM theorem:

$$\frac{\Gamma \Rightarrow \Delta, \forall x(\varphi(x) \leftrightarrow x = t)}{\Gamma \Rightarrow \Delta, t = \imath x\varphi(x)}$$

and

$$\frac{\forall x(\varphi(x) \leftrightarrow x = t), \Gamma \Rightarrow \Delta}{t = \imath x\varphi(x), \Gamma \Rightarrow \Delta}$$

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$$\frac{\varphi(a), \Gamma_1 \Rightarrow \Delta_2, a = t \qquad a = t, \Gamma_2 \Rightarrow \Delta_2, \varphi(a)}{\Gamma \Rightarrow \Delta, t = \imath x \varphi(x)}$$

where a is not in Γ, Δ, φ; and

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Both rules satisfy subformula property and are reductive in the process of cut elimination.

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Reductivity is lost if one premiss is obtained via rule for DD and the second via rule for =. Schematically:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1 \ \ldots \ \Gamma_k \Rightarrow \Delta_k}{\Gamma \Rightarrow \Delta, d = t} \qquad \dfrac{\Pi_1 \Rightarrow \Sigma_1 \ \ldots \ \Pi_n \Rightarrow \Sigma_n}{d = t, \Pi \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \ (Cut)$$

Rule-maker theorem – special case LL:

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$$(2=)\quad \frac{\varphi[x/\tau_2], \Gamma \Rightarrow \Delta}{\tau_1 = \tau_2, \varphi[x/\tau_1], \Gamma \Rightarrow \Delta}$$
Negri and von Plato 2001

$$(3=)\quad \frac{\Gamma \Rightarrow \Delta, \varphi[x/\tau_1]}{\tau_1 = \tau_2, \Gamma \Rightarrow \Delta, \varphi[x/\tau_2]}$$
Manzano 2005

$$(4=)\quad \frac{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2}{\varphi[x/\tau_1], \Gamma \Rightarrow \Delta, \varphi[x/\tau_2]}$$
Reeves 1987

$$(5=)\quad \frac{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2 \qquad \Pi \Rightarrow \Sigma, \varphi[x/\tau_1]}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi[x/\tau_2]}$$
Indrzejczak 2019

$$(6=)\quad \frac{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2 \qquad \varphi[x/\tau_2], \Pi \Rightarrow \Sigma}{\varphi[x/\tau_1], \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$
Baaz and Leitsch 2011

$$(7=)\quad \frac{\Gamma \Rightarrow \Delta, \varphi[x/\tau_1] \qquad \varphi[x/\tau_2], \Pi \Rightarrow \Sigma}{\tau_1 = \tau_2, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$
Nagashima 1966

$$(8=)\quad \frac{\Gamma \Rightarrow \Delta, \tau_1 = \tau_2 \qquad \Pi \Rightarrow \Sigma, \varphi[x/\tau_1] \qquad \varphi[x/\tau_2], \Lambda \Rightarrow \Theta}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta}$$
Indrzejczak 2018

SEQUENT CALCULUS G

Structural rules:

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(AX) $\varphi \Rightarrow \varphi$

(Cut) $$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

(W⇒) $$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

(⇒W) $$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

(C⇒) $$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

(⇒C) $$\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

SEQUENT CALCULUS G

Logical rules:

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Logical rules:

$(\neg\Rightarrow)\ \dfrac{\Gamma\Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma\Rightarrow \Delta}$ $(\Rightarrow\neg)\ \dfrac{\varphi, \Gamma\Rightarrow \Delta}{\Gamma\Rightarrow \Delta, \neg\varphi}$

$(\wedge\Rightarrow)\ \dfrac{\varphi, \psi, \Gamma\Rightarrow \Delta}{\varphi\wedge\psi, \Gamma\Rightarrow \Delta}$ $(\Rightarrow\wedge)\ \dfrac{\Gamma\Rightarrow \Delta, \varphi \quad \Pi\Rightarrow \Sigma, \psi}{\Gamma, \Pi\Rightarrow \Delta, \Sigma, \varphi\wedge\psi}$

$(\vee\Rightarrow)\ \dfrac{\varphi, \Gamma\Rightarrow \Delta \quad \psi, \Pi\Rightarrow \Sigma}{\varphi\vee\psi, \Gamma, \Pi\Rightarrow \Delta, \Sigma}$ $(\Rightarrow\vee)\ \dfrac{\Gamma\Rightarrow \Delta, \varphi, \psi}{\Gamma\Rightarrow \Delta, \varphi\vee\psi}$

$(\rightarrow\Rightarrow)\ \dfrac{\Gamma\Rightarrow \Delta, \varphi \quad \psi, \Pi\Rightarrow \Sigma}{\varphi\rightarrow\psi, \Gamma, \Pi\Rightarrow \Delta, \Sigma}$ $(\Rightarrow\rightarrow)\ \dfrac{\varphi, \Gamma\Rightarrow \Delta, \psi}{\Gamma\Rightarrow \Delta, \varphi\rightarrow\psi}$

$(\leftrightarrow\Rightarrow)\ \dfrac{\Gamma\Rightarrow \Delta, \varphi, \psi \quad \varphi, \psi, \Pi\Rightarrow \Sigma}{\varphi\leftrightarrow\psi, \Gamma, \Pi\Rightarrow \Delta, \Sigma}$ $(\Rightarrow\leftrightarrow)\ \dfrac{\varphi, \Gamma\Rightarrow \Delta, \psi \quad \psi, \Pi \Rightarrow \Sigma, \varphi}{\Gamma, \Pi\Rightarrow \Delta, \Sigma, \varphi\leftrightarrow\psi}$

Quantifier rules:

SEQUENT CALCULUS G

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$(\forall\Rightarrow)\ \dfrac{\varphi[x/b], \Gamma \Rightarrow \Delta}{\forall x\varphi, \Gamma \Rightarrow \Delta}$ $(\Rightarrow\forall)^1\ \dfrac{\Gamma \Rightarrow \Delta, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \forall x\varphi}$

$(\exists\Rightarrow)^1\ \dfrac{\varphi[x/a], \Gamma \Rightarrow \Delta}{\exists x\varphi, \Gamma \Rightarrow \Delta}$ $(\Rightarrow\exists)\ \dfrac{\Gamma \Rightarrow \Delta, \varphi[x/b]}{\Gamma \Rightarrow \Delta, \exists x\varphi}$

SEQUENT CALCULUS G

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side conditions:

1.a is not in Γ, Δ and φ (b is arbitrary).

SEQUENT CALCULUS GRDD1

identity, strictness and description rules:

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$$(=+)\quad \frac{\varphi[x/b], \Gamma \Rightarrow \Delta}{a=b, \varphi[x/a], \Gamma \Rightarrow \Delta} \qquad (=-)\quad \frac{b=b, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

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SEQUENT CALCULUS GRDD1

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$$(Str)\quad \frac{a = d, \Gamma \Rightarrow \Delta}{\varphi[x/d], \Gamma \Rightarrow \Delta} \qquad (Str_=)\quad \frac{a = d_i, \Gamma \Rightarrow \Delta}{d_1 = d_2, \Gamma \Rightarrow \Delta}$$

where φ is atomic but contains at least one occurrence of d and $i \in \{1, 2\}$ in $(Str_=)$; a is not in Γ, Δ, φ.

SEQUENT CALCULUS GRDD1

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where φ is atomic but contains at least one occurrence of d and $i \in \{1,2\}$ in $(Str_=)$; a is not in Γ, Δ, φ.

$$(\imath \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/b], c \approx b \qquad c \approx b, \varphi[x/b], \Gamma \Rightarrow \Delta}{c \approx \imath x \varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \imath)\ \frac{c \approx a, \Gamma \Rightarrow \Delta, \varphi[x/a] \qquad \varphi[x/a], \Gamma \Rightarrow \Delta, c \approx a}{\Gamma \Rightarrow \Delta, c \approx \imath x \varphi}$$

where a is not in Γ, Δ, φ, $c \approx a$ means $c=a$ or $a=c$.

SEQUENT CALCULUS GRDD1

Problem – the cut elimination does not hold for GRDD1:

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$$\dfrac{(\Rightarrow \imath)\ \dfrac{\varphi[x/a], \Gamma_1 \Rightarrow \Delta_1, a = c \qquad a = c, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a]}{\Gamma \Rightarrow \Delta, c = \imath x\varphi} \qquad \dfrac{\psi[x/\imath x\varphi], \Pi \Rightarrow \Sigma}{c = \imath x\varphi, \psi[x/c], \Pi \Rightarrow \Sigma}}{\psi[x/c], \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (Cut)$$

SEQUENT CALCULUS GRDD1

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$$
(Cut)\ \dfrac{(\Rightarrow \imath)\ \dfrac{\varphi[x/a], \Gamma_1 \Rightarrow \Delta_1, a = c \qquad a = c, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a]}{\Gamma \Rightarrow \Delta, c = \imath x\varphi} \qquad \dfrac{\psi[x/\imath x\varphi], \Pi \Rightarrow \Sigma}{c = \imath x\varphi, \psi[x/c], \Pi \Rightarrow \Sigma}}{\psi[x/c], \Gamma, \Pi \Rightarrow \Delta, \Sigma}
$$

$$
(C\Rightarrow)\ \dfrac{(Cut)\ \dfrac{(\Rightarrow \imath)\ \dfrac{(\forall\Rightarrow)\ \dfrac{(\to\Rightarrow)\ \dfrac{Ab \Rightarrow Ab \qquad b = a \Rightarrow b = a}{Ab, Ab \to b = a \Rightarrow b = a}}{Ab, \forall x(Ax \to x = a) \Rightarrow b = a} \qquad \dfrac{Ab \Rightarrow Ab}{b = a, Aa \Rightarrow Ab}}{Aa, \forall x(Ax \to x = a) \Rightarrow a = \imath xAx} \qquad \dfrac{A\imath xAx \Rightarrow A\imath xAx}{a = \imath xAx, Aa \Rightarrow A\imath xAx}}{Aa, Aa, \forall x(Ax \to x = a) \Rightarrow A\imath xAx}}{Aa, \forall x(Ax \to x = a) \Rightarrow A\imath xAx}
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SEQUENT CALCULUS GRDD1

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(LL) may be represented by other rules. One way out is to change $(=+)$ for:

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Cut elimination holds but the subformula property fails for the variant of GRDD1 with $(=)$.

Sequent Calculus GRDD2

Sequent Calculus GRDD2

$$(R)\ \frac{b = b, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad (E)\ \frac{b_1 = b_2, \Gamma \Rightarrow \Delta}{c = b_1, c = b_2, \Gamma \Rightarrow \Delta} \quad (L)\ \frac{\varphi[x/b], \Gamma \Rightarrow \Delta}{c = b, \varphi[x/c], \Gamma \Rightarrow \Delta}$$

$$(\imath \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/b], c \approx b \qquad c \approx b, \varphi[x/b], \Gamma \Rightarrow \Delta}{c \approx \imath x \varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \imath)\ \frac{c \approx a, \Gamma \Rightarrow \Delta, \varphi[x/a] \qquad \varphi[x/a], \Gamma \Rightarrow \Delta, c \approx a}{\Gamma \Rightarrow \Delta, c \approx \imath x \varphi}$$

$$(= \Rightarrow)\ \frac{a_1 = d_1, a_2 = d_2, a_1 = a_2, \Gamma \Rightarrow \Delta}{d_1 = d_2, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow =)\ \frac{\Gamma \Rightarrow \Delta, b_1 = d_1 \qquad \Gamma \Rightarrow \Delta, b_2 = d_2 \qquad \Gamma \Rightarrow \Delta, b_1 = b_2}{\Gamma \Rightarrow \Delta, d_1 = d_2}$$

$$(Str \Rightarrow)\ \frac{a_1 = d_1, \ldots, a_n = d_n, \varphi[x_1/a_1 \ldots x_n/a_n], \Gamma \Rightarrow \Delta}{\varphi[x_1/d_1 \ldots x_n/d_n], \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow Str)\ \frac{\Gamma \Rightarrow \Delta, b_1 = d_1 \qquad \ldots \qquad \Gamma \Rightarrow \Delta, b_n = d_n \qquad \Gamma \Rightarrow \Delta, \varphi[x_1/b_1 \ldots x_n/b_n]}{\Gamma \Rightarrow \Delta, \varphi[x_1/d_1 \ldots x_n/d_n]}$$

where: a, a_1, a_2 are different eigenvariables; φ in (L) is a proper atomic formula; $c \approx t$ in rules for $\imath$ means that either $c = t$ or $t = c$; $\varphi[x_1/d_1 \ldots x_n/d_n]$ in $(Str \Rightarrow)$ and $(\Rightarrow Str)$ is a proper quasi-atomic formula with exactly $n \geq 1$ different definite descriptions as arguments and $\varphi[x_1/b_1 \ldots x_n/b_n]$ is a proper atomic formula

Sequent Calculus GRDD2

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Sequent Calculus GRDD2

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1. if φ is a proper atomic formula, it can be active only in the antecedent, due to (L);
2. if φ is a proper q-atomic, it can be active on both sides but only via $(Str \Rightarrow)$ and $(\Rightarrow Str)$;
3. if it is a d-identity, it can be active on both sides but only via $(=\Rightarrow)$ and $(\Rightarrow=)$;
4. if it is a mixed-identity, it can be active on both sides but only via $(\Rightarrow \imath)$ and $(\imath \Rightarrow)$;
5. if it is an atomic identity, it can be active only in the antecedent, due to (E).

Sequent Calculus GRDD2

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If an atomic formula is a cut formula, then in cases 1 and 5, its occurrence in the left premiss of cut is always parametric $\Longrightarrow$ height-reduction on the left premiss of cut is always possible.

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If an atomic formula is a cut formula, then in cases 1 and 5, its occurrence in the left premiss of cut is always parametric $\Longrightarrow$ height-reduction on the left premiss of cut is always possible.
In cases 2–4, if in at least one premiss of cut it is also parametric, again height-reduction is applicable. If in both premisses of cut, it is the principal formula rank-reduction is applicable.

Sequent Calculus GRDD2

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Sequent Calculus GRDD2

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3. If $\vdash \Gamma \Rightarrow \Delta$ has a cut-free proof, then it is constructed from subformulae of formulae occcurring in Γ, Δ.
4. If $\vdash \Gamma \Rightarrow \Delta$ has a cut-free proof, then the only terms occuring in it are those already occuring in Γ, Δ and eigenvariables.

Interpolation Theorem

Craig and Maehara:

Interpolation Theorem

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logic L satisfies the *Craig interpolation property* if, for all formulae φ and ψ, if $\models \varphi \rightarrow \psi$, then there is an interpolant χ such that $\models \varphi \rightarrow \chi$, $\models \chi \rightarrow \psi$ and χ only contains non-logical symbols common to both φ and ψ. Moreover, if φ and ψ have no non-logical symbols in common, then either $\neg\varphi$ is valid or ψ is valid.

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If $\vdash \Gamma \Rightarrow \Delta$, then for any partition $((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2))$ we can find φ, such that:

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Maehara's theorem implies Craig's theorem.

Interpolation for GRDD2

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PROOF: By induction on the height of a proof of $\Gamma \Rightarrow \Delta$ in cut-free GRDD2.

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The basis (case $k = 0$) and inductive cases for connectives and quantifiers as in classical FOL.

Interpolation for GRDD2; the case of (R):

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If b is not present in φ, then φ satisfies also conditions for being an interpolant of the conclusion; provability and language conditions are trivially satisfied.

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Otherwise we must consider if $b \in L(\Gamma_1 \sqcup \Delta_1)$ or not.
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In the second it has the form $\forall x \varphi[b/x]$ which guarantees the satisfaction of the language condition. Provability conditions are satisfied by applications of $(\Rightarrow \forall)$ to $\Gamma_1 \Rightarrow \Delta_1, \varphi$ (correct since b is only in φ) and $(\forall \Rightarrow)$ to $\varphi, \Gamma_2 \Rightarrow \Delta_2$.

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We could equally well consider an arbitrary partition of the premiss with $b = b$ belonging to (Γ_2, Δ_2) and an interpolant φ with the same result, i.e. the interpolant for the conclusion being either φ or $\exists x \varphi[b/x]$ (if b is in φ but not in Γ_2, Δ_2).

Interpolation Theorem

Interpolation for GRDD2; the case of $(\Longrightarrow)$ and $(Str \Rightarrow)$:

Interpolation Theorem

Interpolation for GRDD2; the case of ($\Longrightarrow$) and ($Str \Rightarrow$):

In the case of ($\Longrightarrow$) and ($Str \Rightarrow$) the situation is even simpler. For any partition of the conclusion, an interpolant φ is inherited from the respective partition of the premiss. It means that if the principal formula belongs to the left (right) division, then we take the same partition of Γ, Δ with all side formulae belonging to the left (right) division. Note that in this case $a_1, ..., a_n$ are eigenvariables so it is not possible that any of them belongs to φ and we do not need any quantification on φ to satisfy the language condition.

Interpolation for GRDD2, the case of $(\Rightarrow \imath)$

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Interpolation Theorem

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In the case of $(\Rightarrow \imath)$, for arbitrary partition, $c \approx \imath x\varphi$ is either in the left or in the right division. In the first case, by the IH we have:

1. $\vdash c \approx a, \Gamma_1 \Rightarrow \Delta_1, \varphi[x/a], \psi_1$
2. $\vdash \psi_1, \Gamma_2 \Rightarrow \Delta_2$
3. $L(\psi_1) \subseteq L(\{c \approx a, \varphi[x/a]\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$
4. $\vdash \varphi[x/a], \Gamma_1 \Rightarrow \Delta_1, c \approx a, \psi_2$
5. $\vdash \psi_2, \Gamma_2 \Rightarrow \Delta_2$
6. $L(\psi_2) \subseteq L(\{c \approx a, \varphi[x/a]\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$

with ψ_1, ψ_2 being interpolants of these partitions of the premisses.

Interpolation Theorem

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We obtain the interpolant $\psi_1 \vee \psi_2$ of the conclusion in the left division since it holds by 1, 4 and 2, 5:

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$$(\vee \Rightarrow)\,\dfrac{\psi_1, \Gamma_2 \Rightarrow \Delta_2 \qquad \psi_2, \Gamma_2 \Rightarrow \Delta_2}{\psi_1 \vee \psi_2, \Gamma_2 \Rightarrow \Delta_2}$$

The language condition
$L(\psi_1 \vee \psi_2) \subseteq L(\{c \approx \imath x\varphi[x/a]\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$ is satisfied by 3, 4 and the fact that a is an eigenvariable.

Interpolation Theorem

Interpolation for GRDD2, the case of $(\Rightarrow \imath)$

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Let $c \approx \imath x \varphi$ be in the right division of arbitrary partition, then by the IH we have:

1. $\vdash \Gamma_1 \Rightarrow \Delta_1, \psi_1$
2. $\vdash \psi_1, c \approx a, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a]$
3. $L(\psi_1) \subseteq L(\Gamma_1 \sqcup \Delta_1) \cap L(\{c \approx a, \varphi[x/a]\} \sqcup \Gamma_2 \sqcup \Delta_2)$
4. $\vdash \Gamma_1 \Rightarrow \Delta_1, \psi_2$
5. $\vdash \psi_2, \varphi[x/a], \Gamma_2 \Rightarrow \Delta_2, c \approx a$
6. $L(\psi_2) \subseteq L(\Gamma_1 \sqcup \Delta_1) \cap L(\{c \approx a, \varphi[x/a]\} \sqcup \Gamma_2 \sqcup \Delta_2)$

Interpolation for GRDD2, the case of $(\Rightarrow \imath)$

Let $c \approx \imath x\varphi$ be in the right division of arbitrary partition, then by the IH we have:

1. $\vdash \Gamma_1 \Rightarrow \Delta_1, \psi_1$
2. $\vdash \psi_1, c \approx a, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a]$
3. $L(\psi_1) \subseteq L(\Gamma_1 \sqcup \Delta_1) \cap L(\{c \approx a, \varphi[x/a]\} \sqcup \Gamma_2 \sqcup \Delta_2)$
4. $\vdash \Gamma_1 \Rightarrow \Delta_1, \psi_2$
5. $\vdash \psi_2, \varphi[x/a], \Gamma_2 \Rightarrow \Delta_2, c \approx a$
6. $L(\psi_2) \subseteq L(\Gamma_1 \sqcup \Delta_1) \cap L(\{c \approx a, \varphi[x/a]\} \sqcup \Gamma_2 \sqcup \Delta_2)$

and the interpolant for the conclusion is $\psi_1 \wedge \psi_2$ by symmetric argument.

Interpolation for GRDD2, the case of $(\imath \Rightarrow)$

Interpolation Theorem

Interpolation for GRDD2, the case of $(\imath \Rightarrow)$

The situation with computing an interpolant for $(\imath \Rightarrow)$ is similar, i.e. with interpolants being basically either $\psi_1 \vee \psi_2$ or $\psi_1 \wedge \psi_2$, but with an important proviso. Since b is an arbitrary parameter it is possible, in case of the left division of the principal formula, that b is in $\psi_1 \vee \psi_2$ but not in Γ_1, Δ_1. Then the interpolant for the conclusion must be $\forall x(\psi_1 \vee \psi_2)[b/x]$. Similarly, in case of the right division, if b occurs in $\psi_1 \wedge \psi_2$ but not in Γ_2, Δ_2, then it has the form $\exists x(\psi_1 \wedge \psi_2)[b/x]$. All other details of the reasoning remain the same.

Interpolation Theorem

Interpolation for GRDD2, the case of $(\Rightarrow =)$:

Interpolation for GRDD2, the case of $(\Rightarrow=)$:

In case of the left division, by the IH we have:

1. $\vdash \Gamma_1 \Rightarrow \Delta_1, b_1 = d_1, \varphi_1$
2. $\vdash \varphi_1, \Gamma_2 \Rightarrow \Delta_2$
3. $L(\varphi_1) \subseteq L(\{b_1 = d_1\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$
4. $\vdash \Gamma_1 \Rightarrow \Delta_1, b_2 = d_2, \varphi_2$
5. $\vdash \varphi_2, \Gamma_2 \Rightarrow \Delta_2$
6. $L(\varphi_2) \subseteq L(\{b_2 = d_2\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$
7. $\vdash \Gamma_1 \Rightarrow \Delta_1, b_1 = b_2, \varphi_3$
8. $\vdash \varphi_3, \Gamma_2 \Rightarrow \Delta_2$
9. $L(\varphi_3) \subseteq L(\{b_1 = b_2\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$

Interpolation for GRDD2, the case of ($\Rightarrow=$):

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Either $b_1, b_2 \in L(\Gamma_1 \sqcup \Delta_1)$ or not.
Assume the former, then the interpolant is $\varphi_1 \vee \varphi_2 \vee \varphi_3$. The following proofs justify provability conditions on the basis of 1, 4, 7, and 2, 5, 8:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1, b_1 = d_1, \varphi_1 \qquad \Gamma_1 \Rightarrow \Delta_1, b_2 = d_2, \varphi_2 \qquad \Gamma_1 \Rightarrow \Delta_1, b_1 = b_2, \varphi_3}{\Gamma_1 \Rightarrow \Delta_1, d_1 = d_2, \varphi_1, \varphi_2, \varphi_3} (\Rightarrow=)}{\Gamma_1 \Rightarrow \Delta_1, d_1 = d_2, \varphi_1 \vee \varphi_2 \vee \varphi_3} (\Rightarrow \vee)$$

$$(\vee \Rightarrow) \dfrac{(\vee \Rightarrow) \dfrac{\varphi_1, \Gamma_2 \Rightarrow \Delta_2 \qquad \varphi_2, \Gamma_2 \Rightarrow \Delta_2}{\varphi_1 \vee \varphi_2, \Gamma_2 \Rightarrow \Delta_2} \qquad \varphi_3, \Gamma_2 \Rightarrow \Delta_2}{\varphi_1 \vee \varphi_2 \vee \varphi_3, \Gamma_2 \Rightarrow \Delta_2}$$

$L(\varphi_1 \vee \varphi_2 \vee \varphi_3) \subseteq L(\{d_1 = d_2\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$ follows from 3, 6, 9.

Interpolation Theorem

Interpolation for GRDD2, the case of $(\Rightarrow=)$:

Either $b_1, b_2 \in L(\Gamma_1 \sqcup \Delta_1)$ or not.
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$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1, b_1 = d_1, \varphi_1 \qquad \Gamma_1 \Rightarrow \Delta_1, b_2 = d_2, \varphi_2 \qquad \Gamma_1 \Rightarrow \Delta_1, b_1 = b_2, \varphi_3}{\Gamma_1 \Rightarrow \Delta_1, d_1 = d_2, \varphi_1, \varphi_2, \varphi_3}\,(\Rightarrow=)}{\Gamma_1 \Rightarrow \Delta_1, d_1 = d_2, \varphi_1 \vee \varphi_2 \vee \varphi_3}\,(\Rightarrow \vee)$$

$$(\vee\Rightarrow)\,\dfrac{(\vee\Rightarrow)\,\dfrac{\varphi_1, \Gamma_2 \Rightarrow \Delta_2 \qquad \varphi_2, \Gamma_2 \Rightarrow \Delta_2}{\varphi_1 \vee \varphi_2, \Gamma_2 \Rightarrow \Delta_2} \qquad \varphi_3, \Gamma_2 \Rightarrow \Delta_2}{\varphi_1 \vee \varphi_2 \vee \varphi_3, \Gamma_2 \Rightarrow \Delta_2}$$

$L(\varphi_1 \vee \varphi_2 \vee \varphi_3) \subseteq L(\{d_1 = d_2\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$ follows from 3, 6, 9.

In case some of b_1, b_2 (or both) are not present in Γ_1, Δ_1 but occur in interpolants of premisses, an additional quantification is required. Accordingly interpolants for the conclusion should have the form: $\forall x(\varphi_1 \vee \varphi_2 \vee \varphi_3)[x/b_1]$ or $\forall x(\varphi_1 \vee \varphi_2 \vee \varphi_3)[x/b_2]$ or $\forall xy(\varphi_1 \vee \varphi_2 \vee \varphi_3)[x/b_1, y/b_2]$.

Interpolation for GRDD2, the case of ($\Rightarrow=$):

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Interpolation for GRDD2, the case of ($\Rightarrow Str$):

Interpolation Theorem

Interpolation for GRDD2, the case of ($\Rightarrow Str$):

There is nothing essentially new in the procedure of computing interpolants for the conclusion of ($\Rightarrow Str$). For n different descriptions in the principal formula we have, by the IH, $n+1$ interpolants for arbitrary partitions of the premisses. Therefore, in case of the left division our interpolant obtains one of the form $\psi_1 \vee \ldots \psi_{n+1}, \ldots, \forall x_1 \ldots x_{n+1}(\psi_1 \vee \ldots \psi_{n+1})[b_1/x_1, \ldots, b_{n+1}/x_{n+1}]$ depending on the occurrences of b_i in Γ_1, Δ_1 and interpolants. Dually, in case of the right division, suitable interpolants are conjunctions of $\psi_1, \ldots, \psi_{n+1}$, possibly existentially quantified.

Interpolation for GRDD2, the case of (E) and (L):

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In both rules there are two principal formulae, thus there are four possible classes of partitions of the conclusion. For (E) they are:

(a) $(c = b_1, c = b_2, \Gamma_1, \Delta_1), (\Gamma_2, \Delta_2)$

(b) $(\Gamma_1, \Delta_1), (c = b_1, c = b_2, \Gamma_2, \Delta_2)$

(c) $(c = b_1, \Gamma_1, \Delta_1), (c = b_2, \Gamma_2, \Delta_2)$

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(d) $(c = b_2, \Gamma_1, \Delta_1), (c = b_1, \Gamma_2, \Delta_2)$.

In case of the premiss there are only two possible cases, so by the IH we have:

1.1. $\vdash b_1 = b_2, \Gamma_1 \Rightarrow \Delta_1, \varphi$

1.2. $\vdash \varphi, \Gamma_2 \Rightarrow \Delta_2$

1.3. $L(\varphi) \subseteq L(\{b_1 = b_2\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$

and

2.1. $\vdash \Gamma_1 \Rightarrow \Delta_1, \varphi$

2.2. $\vdash \varphi, b_1 = b_2, \Gamma_2 \Rightarrow \Delta_2$

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In both rules there are two principal formulae, thus there are four possible classes of partitions of the conclusion. For (E) they are:

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2.2. $\vdash \varphi, b_1 = b_2, \Gamma_2 \Rightarrow \Delta_2$

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For the case (a) from 1.1 we derive $\vdash c = b_1, c = b_2, \Gamma_1 \Rightarrow \Delta_1, \varphi$ by (E), and 1.3. obviously implies $L(\varphi) \subseteq L(\{c = b_1, c = b_2\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$, hence φ satisfies the conditions for being an interpolant and we are done.

Interpolation Theorem

Interpolation for GRDD2, the case of (E)

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By symmetric argument but on the basis of 2.1., 2.2. and 2.3. we prove that in the case (b) the interpolant is also inherited from the respective partition of the premiss.

Interpolation for GRDD2, the case of (E)

By symmetric argument but on the basis of 2.1., 2.2. and 2.3. we prove that in the case (b) the interpolant is also inherited from the respective partition of the premiss. Consider the case (c). By the IH, 1.1., 1.2., 1.3 hold and either $b_2 \in L(\Gamma_1 \sqcup \Delta_1)$ or not.

Interpolation Theorem

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By symmetric argument but on the basis of 2.1., 2.2. and 2.3. we prove that in the case (b) the interpolant is also inherited from the respective partition of the premiss. Consider the case (c). By the IH, 1.1., 1.2., 1.3 hold and either $b_2 \in L(\Gamma_1 \sqcup \Delta_1)$ or not. Assume the former, then the interpolant is $c = b_2 \to \varphi$ and the provability conditions are satisfied by the following arguments on the basis of 1.1. and 1.2.:

$$\dfrac{\dfrac{b_1 = b_2, \Gamma_1 \Rightarrow \Delta_1, \varphi}{c = b_1, c = b_2, \Gamma_1 \Rightarrow \Delta_1, \varphi}\,(E)}{c = b_1, \Gamma_1 \Rightarrow \Delta_1, c = b_2 \to \varphi}\,(\Rightarrow\to)$$

$$(\to\Rightarrow)\,\dfrac{c = b_2, \Gamma_2 \Rightarrow \Delta_2, c = b_2 \qquad \varphi, \Gamma_2 \Rightarrow \Delta_2}{c = b_2 \to \varphi, c = b_2, \Gamma_2 \Rightarrow \Delta_2}$$

Note that $L(c = b_2 \to \varphi) \subseteq L(\{c = b_1\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\{c = b_2\} \sqcup \Gamma_2 \sqcup \Delta_2)$ follows by 1.3. and the fact that c is in both parts and b_2 as well by the assumption.

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$$\dfrac{\dfrac{b_1 = b_2, \Gamma_1 \Rightarrow \Delta_1, \varphi}{c = b_1, c = b_2, \Gamma_1 \Rightarrow \Delta_1, \varphi}\,(E)}{c = b_1, \Gamma_1 \Rightarrow \Delta_1, c = b_2 \to \varphi}\,(\Rightarrow\to)$$

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Note that $L(c = b_2 \to \varphi) \subseteq L(\{c = b_1\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\{c = b_2\} \sqcup \Gamma_2 \sqcup \Delta_2)$ follows by 1.3. and the fact that c is in both parts and b_2 as well by the assumption.

Let $b_2 \notin L(\Gamma_1 \sqcup \Delta_1)$. If b_2 is not in φ keep the interpolant intact, but if b_2 occurs in φ, it has the form $\forall x(c = x \to \varphi[b_2/x])$. Provability conditions are satisfied by the application of $(\Rightarrow \forall)$ to the interpolant in the first proof figure (correct since b_2 occurs only in the interpolant by assumption) and by $(\forall \Rightarrow)$ in the second.

Interpolation for GRDD2, the case of (E)

Interpolation for GRDD2, the case of (E)

Now consider the fourth case, i.e. (d). By the IH, 2.1., 2.2., 2.3. hold and again, either $b_2 \in L(\Gamma_2 \sqcup \Delta_2)$ or not.

Interpolation for GRDD2, the case of (E)

Now consider the fourth case, i.e. (d). By the IH, 2.1., 2.2., 2.3. hold and again, either $b_2 \in L(\Gamma_2 \sqcup \Delta_2)$ or not. In the first case the interpolant is $\varphi \wedge c = b_2$ and in the second $\exists x(\varphi[b_2/x] \wedge c = x)$ (provided that b_2 is also in φ). The respective proof figures are the following:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad c = b_2, \Gamma_1 \Rightarrow \Delta_1, c = b_2}{c = b_2, \Gamma_1 \Rightarrow \Delta_1, \varphi \wedge c = b_2}\ (\Rightarrow \wedge)}{c = b_2, \Gamma_1 \Rightarrow \Delta_1, \exists x(\varphi[b_2/x] \wedge c = x)}\ (\Rightarrow \exists)$$

$$(\exists \Rightarrow)\ \dfrac{(\wedge \Rightarrow)\ \dfrac{(E)\ \dfrac{\varphi, b_1 = b_2, \Gamma_2 \Rightarrow \Delta_2}{\varphi, c = b_1, c = b_2, \Gamma_2 \Rightarrow \Delta_2}}{\varphi \wedge c = b_2, c = b_1, \Gamma_2 \Rightarrow \Delta_2}}{\exists x(\varphi[b_2/x] \wedge c = x), c = b_1, \Gamma_2 \Rightarrow \Delta_2}$$

with the last applications of rules for $\exists$ not required if $b_2 \in L(\Gamma_2 \sqcup \Delta_2)$ or does not occur in φ.

Interpolation Theorem

Interpolation for GRDD2, the case of (E)

Now consider the fourth case, i.e. (d). By the IH, 2.1., 2.2., 2.3. hold and again, either $b_2 \in L(\Gamma_2 \sqcup \Delta_2)$ or not. In the first case the interpolant is $\varphi \wedge c = b_2$ and in the second $\exists x(\varphi[b_2/x] \wedge c = x)$ (provided that b_2 is also in φ). The respective proof figures are the following:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad c = b_2, \Gamma_1 \Rightarrow \Delta_1, c = b_2}{c = b_2, \Gamma_1 \Rightarrow \Delta_1, \varphi \wedge c = b_2}\ (\Rightarrow \wedge)}{c = b_2, \Gamma_1 \Rightarrow \Delta_1, \exists x(\varphi[b_2/x] \wedge c = x)}\ (\Rightarrow \exists)$$

$$(\exists \Rightarrow)\ \dfrac{(\wedge \Rightarrow)\ \dfrac{(E)\ \dfrac{\varphi, b_1 = b_2, \Gamma_2 \Rightarrow \Delta_2}{\varphi, c = b_1, c = b_2, \Gamma_2 \Rightarrow \Delta_2}}{\varphi \wedge c = b_2, c = b_1, \Gamma_2 \Rightarrow \Delta_2}}{\exists x(\varphi[b_2/x] \wedge c = x), c = b_1, \Gamma_2 \Rightarrow \Delta_2}$$

with the last applications of rules for $\exists$ not required if $b_2 \in L(\Gamma_2 \sqcup \Delta_2)$ or does not occur in φ.

In the latter case, i.e. if $b_2 \notin L(\Gamma_2 \sqcup \Delta_2)$ but b_2 occurs in φ, the quantification of b_2 is necessary to satisfy the language condition.

Interpolation for GRDD2, the case of (L)

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In the case of (L) the situation is similar but simpler. Again there are four possible classes of partitions of the conclusion:

(a) $(c = b, \varphi[x/c], \Gamma_1, \Delta_1), (\Gamma_2, \Delta_2)$

(b) $(\Gamma_1, \Delta_1), (c = b, \varphi[x/c], \Gamma_2, \Delta_2)$

(c) $(c = b, \Gamma_1, \Delta_1), (\varphi[x/c], \Gamma_2, \Delta_2)$

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and by the IH we have:

1.1. $\vdash \varphi[x/b], \Gamma_1 \Rightarrow \Delta_1, \psi$

1.2. $\vdash \psi, \Gamma_2 \Rightarrow \Delta_2$

1.3. $L(\psi) \subseteq L(\{\varphi[x/b]\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$

and

2.1. $\vdash \Gamma_1 \Rightarrow \Delta_1, \psi$

2.2. $\vdash \psi, \varphi[x/b], \Gamma_2 \Rightarrow \Delta_2$

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The cases (a) and (b) allow us to inherit the interpolant ψ from the premiss. Cases (c) and (d) require similar proofs like for (E). In the first case the interpolant is $\psi \wedge \varphi[x/c]$ obtained from 2.1., 2.2. in the following way:

$$\dfrac{\Gamma_1 \Rightarrow \Delta_1, \psi \qquad c = b, \Gamma_1 \Rightarrow \Delta_1, c = b}{c = b, \Gamma_1 \Rightarrow \Delta_1, \psi \wedge c = b}\,(\Rightarrow \wedge)$$

$$(\wedge \Rightarrow)\,\dfrac{(L)\,\dfrac{\psi, \varphi[x/b], \Gamma_2 \Rightarrow \Delta_2}{\psi, c = b, \varphi[x/c], \Gamma_2 \Rightarrow \Delta_2}}{\psi \wedge c = b, \varphi[x/c], \Gamma_2 \Rightarrow \Delta_2}$$

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In the case (d) we obtain $c = b \rightarrow \psi$ as an interpolant on the basis of 1.1., 1.2.

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$$(\wedge \Rightarrow) \frac{(L)\dfrac{\psi, \varphi[x/b], \Gamma_2 \Rightarrow \Delta_2}{\psi, c = b, \varphi[x/c], \Gamma_2 \Rightarrow \Delta_2}}{\psi \wedge c = b, \varphi[x/c], \Gamma_2 \Rightarrow \Delta_2}$$

In the case (d) we obtain $c = b \rightarrow \psi$ as an interpolant on the basis of 1.1., 1.2. In both cases the language condition is satisfied without further inference steps. □

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