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A NEW CRITERION OF DECIDABILITY FOR INTERMEDIATE LOGICS

In this paper I study Łukasiewicz-style refutation procedures (cf. [8], also [3]) for intermediate logics as a method of obtaining decidability results. This method proves to be more general than Harrop's criterion (cf. [4]) saying that every finitely axiomatizable logic with the finite model property is decidable. There are $2^{\aleph_0}$ intermediate logics without the f.m.p. (cf. [1], [2]). I give a more general criterion of decidability applicable also to some of those logics.

Notation and definitions

The script capitals $\mathcal{A}, \mathcal{B}, \ldots$ denote countable Heyting algebras (pseudo-Boolean algebras) called just algebras later on. We say that $\mathcal{A}$ is strongly compact iff there is a greatest element in the set $A - \{1_A\}$ (denoted by $*_A$). As usual $E(\mathcal{A})$ is the set of all $\mathcal{A}$-tautologies (wffs true in $\mathcal{A}$).

For denotes the set of all wffs generated from the set of sentential variables Var by the connectives $\neg, \wedge, \vee, \rightarrow$ (negation, conjunction, disjunction, implication). I assume that $Var = \bigcup\{V_i : i \in N\}$, where N is the set of the positive integers, $V_i = \{p_i^k : k \in N\}$, and $V_i \cap V_j = \emptyset$ for all $i \neq j$. By an intermediate logic we mean a proper subset of For containing INT (the theorems of the intuitionistic sentential calculus) and closed under *modus ponens* and substitution. $Var(\alpha)$, $S(\alpha)$ denote the set of the sentential variables of α, and the set of the subformulas of α respectively.

Following Yankov [11] and Wroński [10], for every algebra $\mathcal{A}$ we define the description of $\mathcal{A}$ thus

$DS(\mathcal{A}) = \{p_x \otimes p_y \leftrightarrow p_{x \otimes y} : x, y \in A, \otimes \in \{\wedge, \vee, \rightarrow\}\} \cup \{\neg p_x \leftrightarrow p_{\neg x} : x \in A\}$,

where $p_x \in Var$ $(x \in A)$ and $p_x \neq p_y$ for all $x \neq y$. Moreover, for any algebra $\mathcal{A}$ such that $|A| > 1$, I define the set $RF(\mathcal{A})$ as follows

$$RF(\mathcal{A}) = \{\bigwedge X \rightarrow p : p \in Var(X) - \{p_{1_A}\} \neq \emptyset, X \subseteq DS(\mathcal{A}), |X| < \aleph_0\}.$$

By a refutation formulation of a logic L I mean a set $X \subseteq For - L$ such that for every $\alpha \in For - L$ there is $\beta \in X$ derivable from α by substitution, *modus ponens*, and theorems of L.

First we note the following obvious

PROPOSITION. *If a recursively axiomatizable logic has a recursive refutation formulation then it is decidable.*

LEMMA. *Let $\mathcal{A}$ be an algebra such that $|A| > 1$, and let $\alpha \in For$. Then $\alpha \notin E(\mathcal{A})$ iff $e(\alpha) \rightarrow \beta \in INT$ for some $\beta \in RF(\mathcal{A})$ and some substitution $e : For \rightarrow For$.*

PROOF. ($\Leftarrow$) It is enough to prove that $RF(\mathcal{A}) \cap E(\mathcal{A}) = \emptyset$. Assume that $\beta \in RF(\mathcal{A})$, that is $\beta = \bigwedge X \rightarrow p$ for some finite $X \subseteq DS(\mathcal{A})$ and some $p \in Var(X) - \{p_{1_A}\}$. Let $v : For \rightarrow A$ be a valuation in $\mathcal{A}$ such that $v(p_x) = x$ $(x \in A)$. Then $v(\bigwedge X) = 1_A$ and $v(p) \neq 1_A$. Thus $v(\beta) \neq 1_A$.

($\Rightarrow$) Assume that $\alpha \notin E(\mathcal{A})$. Then $v(\alpha) \neq 1_A$ for some valuation v in $\mathcal{A}$. Let e be a substitution such that $e(p) = p_x$ if $v(p) = x$ $(p \in Var)$, and let

$$X = \{p_{v(\alpha_1)} \otimes p_{v(\alpha_2)} \leftrightarrow p_{v(\alpha_1) \otimes v(\alpha_2)} : \alpha_1 \otimes \alpha_2 \in S(\alpha), \otimes \in \{\wedge, \vee, \rightarrow\}\} \cup$$

$$\cup \{\neg p_{v(\alpha_1)} \leftrightarrow p_{\neg v(\alpha_1)} : \neg \alpha_1 \in S(\alpha)\},$$

and $\beta = \bigwedge X \rightarrow p_{v(\alpha)}$.

Suppose to the contrary that $e(\alpha) \rightarrow \beta \notin INT$. Then there is a strong compact algebra $\mathcal{B}$ (e.g. a suitable Jaśkowski algebra) such that $w(e(\alpha) \rightarrow \beta) = *_B$ for some valuation w in $\mathcal{B}$, that is $w(e(\alpha)) = 1_B$, $w(\bigwedge X) = 1_B$, and $w(p_{v(\alpha)}) \neq 1_B$. Thus it is easy show that for every $\gamma \in S(\alpha)$ we have $w \circ e(\gamma) = w(p_{v(\gamma)})$ (induction on the complexity of γ). Hence $w \circ e(\alpha) = w(p_{v(\alpha)}) \neq 1_B$. Contradiction. Q.E.D.

The following three definitions are after M.O. Rabin [7].

An indexing of a set S is a one-to-one mapping $i : S \to N$ such that $i(S)$ is a recursive subset of N. For every $s \in S$, $i(s)$ will be called the index of s, and for every $j \in i(S)$, s_j will denote the element of S whose index is j.

We say that an indexing i of an algebra $\mathcal{A}$ is admissible iff the functions c, d, k from $i(A) \times i(A)$ into $i(A)$ and the function $n : i(A) \to i(A)$ such that for every $j, m \in i(A)$

$a_j \to a_m = a_{c(j,m)}$
$a_j \vee a_m = a_{d(j,m)}$
$a_j \wedge a_m = a_{k(j,m)}$
$\neg a_j = a_{n(j)}$

are computable functions.

An algebra is said to be computable iff it has an admissible indexing.

Observe that if an algebra $\mathcal{A}$ is computable then the set $DS(\mathcal{A})$ is recursive.

A logic L is said to be complete for a set K of algebras iff $L = \bigcap\{E(\mathcal{A}) : \mathcal{A} \in K\}$.

I say that a logic L has the computable model property iff L is complete for a set K of algebras such that

(i) Every algebra in K is computable.

(ii) K has an indexing.

Theorem. *If a logic L has the c.m.p. then L has a recursive refutation formulation.*

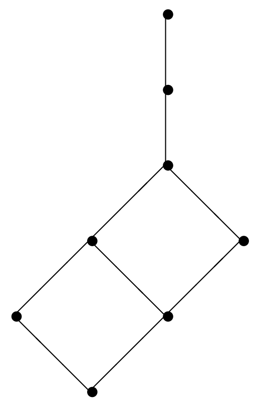
Proof. Assume that L has a c.m.p., that is $L = \bigcap\{E(\mathcal{A}) : \mathcal{A} \in K\}$ for some set K of computable algebras with an indexing $i : K \to N$. This means that $i(K)$ is a recursive subset of N. Let $X = \bigcup\{RF(\mathcal{A}_j) : j \in i(K)\}$, where $\mathcal{A}_j$ is the algebra of K having j for index, and $Var(RF(\mathcal{A}_j)) \subseteq V_j$ $(j \in i(K))$. By Lemma the set X is a refutation formulation of L. Moreover, X is recursive since each set $RF(\mathcal{A}_j)$ is recursive and a formula α can be in X only if $Var(\alpha) \subseteq V_j$ for some $j \in i(K)$. Q.E.D.

Proposition and Theorem yield

Criterion. *If a recursively axiomatizable logic has the computable model property then it is decidable.*

REMARKS (i) Harrop's theorem for intermediate logics is a special case of Criterion since every finitely axiomatizable logic with the f.m.p. has the c.m.p..

(ii) Let $\mathcal{A}$ be the Nishimura lattice (cf. [6]) and $\mathcal{B} =$



Kuznetsov and Gerchiu [5] showed that the logic $E(\mathcal{A} \otimes \mathcal{B})$ is finitely axiomatizable but does not have the f.m.p. ($\otimes$ denotes the Troelstra sum, cf. [9]). However, this logic is decidable in view of Criterion.

References

[1] W. J. Blok, $2^{\aleph_0}$ *varieties of Heyting algebras not generated by their finite members*, **Algebra Universalis**, vol. 7 (1977), pp. 115–117.

[2] W. Dziobiak, *On finite approximability of ψ-intermediate logics*, **Studia Logica**, vol. 41 (1982), pp. 67–73.

[3] D. M. Gabbay, **Semantical Investigations in Heyting's Intuitionistic Logic**, D. Reidel, Dordrecht 1981.

[4] R. Harrop, *On the existence of finite models and decision procedures for propositional calculi*, **Proceedings of Cambridge Philosophical Society**, vol. 54 (1958), pp. 1–13.

[5] A. V. Kuznetzov, V. Y. Gerchiu, *On superintuitionistic logics and finite approximability*, **Doklady AN SSSR**, vol. 195 (1970), pp. 1029–1032.

[6] I. Nishimura, *On formulas of the one variable in intuitionistic propositional calculus*, **Journal of Symbolic Logic**, vol. 25 (1960), pp. 327–331.

[7] M. O. Rabin, *Computable algebra, general theory and theory of computable fields*, **Transactions of the American Mathematical Society**, vol. 95 (1960), pp. 341–360.

[8] J. Słupecki, G. Bryll, U. Wybraniec-Skardowska, *Theory of rejected propositions I*, **Studia Logica**, vol. 29 (1971), pp. 75–123.

[9] A. S. Trolestra, *On intermediate propositional logics*, **Indagationes Mathematicae**, vol. 27 (1965), pp. 141–152.

[10] A. Wroński, *On cardinalities of matrices strongly adequate for the intuitionistic propositional logic*, **Reports of Mathematical Logic**, vol. 3 (1974), pp. 67–72.

[11] V. A. Yankov, *Relationship between deducibility in the intuitionistic propositional logic and finite implicational structures*, **Doklady AN SSSR**, vol. 151 (1963), pp. 1203–1204.

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