Bulletin of the **S**ection of **L**ogic
Volume 19/1 (1990), pp. 15–20

reedition 2005 [original edition, pp. 15–20]

José M. Méndez

URQUHART'S *C* WITH
MINIMAL NEGATION

Introduction

A. Urquhart introduced in [5] a positive propositional logic called *C* as a previous step in defining an algebraic semantics for Łukasiewicz's infinite-valued logic $Ł_\omega$. The logic *C* is of independent interest because, as Urquhart points out, it seems more adequate than $Ł_\omega$ to promising interpretation of multivalent logics, the "multiset interpretation".

This interpretation of multivalent logics immediately connects them with certain relevance logics. Relying on this connection, we have provide in [2] a Routley-Meyer type relational semantics for Urquhart's *C*.

Now, there are (essentially) three possibilities for extending *C* with a negation connective. The first is a kind of semiclassical negation: the result is $Ł_\omega$; the second is a semi-intuitionistic negation: the result is a cousin of Dummet's *LC* (see [3]); and, finally, we can add a "minimal negation" to *C*.

The aim of this paper is to provide a Routley-Meyer type relational semantics for Urquhart's *C* with minimal negation.

1 Urquhart's *C*

Urquhart's *C* can be axiomatized with

Axioms
A1 $(A \to B) \to ((B \to C) \to (A \to C))$

A2 $A \to ((A \to B) \to B)$
A3 $(A\&B) \to A,\ (A\&B) \to B$
A4 $(A \to (B \to (A\&B))$
A5 $A \to (A \vee B),\ B \to (A \vee B)$
A6 $((A \to C)\&(B \to C)) \to ((A \vee B) \to C)$
A7 $(A \to B) \vee (B \to A)$

Rule
Modus Ponens: If $\vdash A$ and $\vdash A \to B$, then $\vdash B$.

2 Semantics

A *model* is the triple $<K, R, \models>$ where K is a set; R is a ternary relation on K subject to the following definitions and postulates for all $a, b, c, d \in K$ with quantifiers ranging over K:

d1. $a \leq b \stackrel{\text{df}}{=} \exists_x Rxab$
d2. $R^2abcd \stackrel{\text{df}}{=} \exists_x(Rabx \text{ and } Rxcd)$
P1. $a \leq a$
P2. $a \leq b$ and $Rbcd \Rightarrow Racd$
P3. $R^2abcd \Rightarrow \exists_x(Racx \text{ and } Rxcd)$
P4. $Rabc \Rightarrow Rbac$
P5. $Rabc$ and $Radc \Rightarrow b \leq c$ or $d \leq c$

Finally, $\models$ is a valuation relation from K to the sentences of C satisfying the following conditions for all $a \in K$:

(i) For each propositional variable p and $a, b \in K, a \models p$ and $a \leq b \Rightarrow b \models p$.
(ii) $a \models A\&B$ iff $a \models A$ and $a \models B$.
(iii) $a \models A \vee B$ iff $a \models A$ or $a \models B$.
(iv) $a \models A \to B$ iff for all $b, c \in K, Rabc$ and $b \models A \Rightarrow c \models B$.

A formula A is a *valid* iff $x \models A$ for all $x \in K$ in all models. We have shown in [2] that A is a theorem of C iff A is valid.

3 C with minimal negation: the logic Cm

To formulate Cm we add to the sentential language of C the propositional constant f. Then, we define $\neg A \stackrel{\text{df}}{=} A \to f$. Now, we note that though

$$A \to \neg\neg A$$
$$(A \to B) \to (\neg B \to \neg A)$$
or $$(A \to \neg B) \to (B \to \neg A)$$

are theorems, such formulas as

$$(A \to \neg A) \to \neg A$$
or $$(A \to B) \to ((A \to \neg B) \to \neg A)$$

are not provable.

4 Semantics for Cm

A *model* is the quadruple $<K, S, R, \models>$ where $<K, R, \models>$ is a model for C and $S \subseteq K$ satisfying the clause

(v) $a \models f$ iff $a \notin S$.

A is *valid* iff $x \models A$ for all $x \in K$ in all models.

5 Completeness of Cm

A theory is a set of formulas of Cm closed under adjunction and provable entailment (that is, a is a *theory* if whenever $A, B \in a$, then $A \& B \in a$; if whenever $A \to B \in Cm$ and $A \in a$, then $B \in a$); a theory a is *null* if no wff belongs to a; *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; *regular* if all theorems of Cm belong to a; finally, a is *consistent* if a does not contain the negation of a theorem of Cm. Now, we define the *canonical model* as the structure $<K_c, S_c, R_c, \models_c>$ where K_c is the set of all non-null prime theories; S_c is a proper subset of K_c formed by all the consistent theories; R_c is defined on K_c as follows: for all formulas A, B and $a, b, c \in K$, $R_c abc$ iff $A \to B \in a$ and $A \in b$, then $B \in c$. Finally, $\models_c$ is defined as follows: for all wff A and $a \in K_c, a \models_c A$ iff $A \in a$.

In what follows we sketch a proof of completeness referring to the results of [4], though the reader might use as well those of [2].

Lemma 1. *If a is non-null theory, then a is regular.*

Proof. Use the theorem $A \to (B \to A)$.

Lemma 2. *If A is not provable in Cm, there is a non-null prime theory T which does not contain A.*

Proof. (Cfr. [4], pp. 307, ff.) "Maximize" on Cm to get a prime theory T without A.

Lemma 3. *Let $< K_c, S_c, R_c, \models_c >$ be the canonical model. For all $a, b \in K_c, a \leq b$ iff $a \subseteq b$.*

Proof. (Cfr. [4], pp. 312-313) Suppose $a \leq b$. Show $a \subseteq b$ using the theorem $A \to A$. Suppose now $a \subseteq b$. It is clear that R_cCmab (a is a theory and $a \subseteq b$). Then, "maximize" Cm to a prime theory x' such that $Rx'ab$. Thus, $a \leq b$.

Lemma 4. *The canonical model is indeed a model.*

Proof. We have to prove that the postulates $P1-P5$ hold in the canonical model and that the canonical valuation relation has properties listed in section 2.

(I) The postulates P1 – P5 hold in the canonical model.

Proof. P1 and P2 are trivial by Lemma 3; P3 and P4 are proved as in [4] p. 313; finally, p5 is easily proved using A7 and Lemma 3

(II) The canonical valuation relation has the desired properties (Cfr. [4], pp. 315. ff.)

Proof. Clause (i) is immediate by Lemma 3; clauses (ii), (iii) are trivial. Thus, the clauses of interest are (iv) and (v).

Clause (iv). $a \models A \to B$ iff for all $b, c \in K_c$, if R_cabc and $b \models_c A$, then $c \models_c B$.

Proof from left to right is trivial. Suppose then $a \not\models_c A \to B$. We show that there are $b', c' \in K_c$ such that $R_cab'c', b' \models_c A$ and $c' \not\models_c B$. Define the non-null theories $b = \{C : A \to C \in Cm\}, c = \{C : \exists D(D \in b$ and $D \to C \in a)\}$. Then, "maximize" c and b to prime theories b', c' such that $R_cab'c', A \in b'$ and $B \notin c'$.

Clause (v). $a \models_c f$ iff $a \notin S_c$.

(a) Suppose that $a \models_c f$. By definition, $f \in a$. Then by the theorem $f \to (A \to f)$ (A is a theorem), $A \to f \in a$. Thus, a is incosistent.

(b) Suppose that a is inconsistent. Then, $A \to f \in a$, A being a theorem. But $R_c axa$ ($P1$ and $P4$) with $x \in K_c$. Thus, $A \in x$ and, so, $f \in a$ ($a \models_c f$).

The proof of clause (v) ends the proof of Lemma 4. Now we prove

THEOREM (COMPLETENESS). *A is a theorem of Cm iff A is valid.*

PROOF. (a) Axioms $A1 - A6$ are proved as usual. $A7$ is proved using $P5$. *Modus Ponens* preserves validity.

(b) If A is not a theorem, then, by Lemma 2, there is a non-null theory T such that $A \notin T$. Therefore, A is invalid by Lemma 4.

6 Adding $\neg$ to C as primitive connective: the logic Cm'

We add to the sentential language of C the unary connective $\neg$ (negation) and the axiom

$$(A \to \neg B) \to (B \to \neg A).$$

The models for Cm' are defined similarly to those for Cm but with

(v') $a \models \neg A$ iff for all $b, c \in S$, not-$Rabc$ or $b \models A$.

instead of clause (v). Then, a similar argument to that developed in [1] shows that Cm and Cm' are, in fact, the same system.

We finish by noting a, we think, remarkable fact. In [3] we have added a "semi-intuitionistic" negation to Urquhart's C to define the system Ci. Semantically the difference between Ci and Cm is, roughly, this: in Ci-models we *have* to require the *consistency* of all members of K_c; in Cm-models, as we have seen, *it suffices* with the *consistency* of *one* member of K_c (e.g., Cm). Now, in the completeness proof of Ci we *do not need* nothing outside of Cm for proving the negative clause of Lemma 4, but it is not so far the positive fragment of the system: the Ci-theorem $\neg A \to (A \to B)$ (unprovable in Cm) has to be used in Lemmas 1-3 and clause (iv) of Lemma 4.

References

[1] José M. Méndez, *Constructive R*, **Bulletin of the Section of Logic**, vol. 16 (1987), pp. 167–175.

[2] José M. Méndez, *Routley-Meyer type semantics for Urquhart's C*, to appear in **Journal of Non-Classical Logic**.

[3] José M. Méndez, *Urquhart's C with semi-intuitionistic negation: Dummet's LC without the contraction axiom*, to appear.

[4] R. Routley and others, **Relevant Logics and their rivals**, Ridgeview, California, 1982.

[5] A. Urquhart, *Many-valued logic*, [in:] D. Gabbay and F. Guenthner (eds) **Handbook of Philosophical Logic**, vol. 3, D. Reidel, Dordrecht-Boston, 1986, pp. 71–116.

Depto de Filosofia y Lógica y Filosofia de la Ciencia
Facultad de Filosofia y CCE
Universitad de Salamanca
P. de Canalejas, 169
37008 Salamanca
Spain