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ON HALLDÉN-COMPLETNESS OF INTERMEDIATE AND MODAL LOGICS

A logic L is said to be Halldén-complete (or Halldén-reasonable) if for any formula $A \vee B$ provable in L, where A and B have no variables in common, $L \vdash A$ or $L \vdash B$.

A. Wroński [4] has obtained the algebraic equivalents of Halldén-completeness for intermediate and modal logics. J. van Benthem and I. Humberstone [2] have given in semantic terms a sufficient condition for Halldén-completeness in normal modal logics; it is unknown whether the condition is necessary.

In this paper we deal with the problem of deciding, given an axiomatization of intermediate logic or normal modal logic containing $S4$, whether the logic is Halldén-complete.

Recently we have shown [1] that the disjunction property of intermediate logics is undecidable. (Recall that an intermediate logic L is said to have disjunction property if $L \vdash A$ or $L \vdash B$ whenever $L \vdash A \vee B$; the definition of disjunction property for modal logics is as follows: $L \vdash A \vee B \Rightarrow L \vdash A$ or $L \vdash B$.)

It is obvious that for intermediate logics the disjunction property implies Halldén-completeness (compare with Theorem 6 below). With the help of this fact we have proved in [1] the following

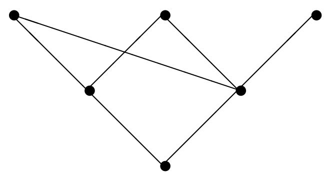
Theorem 1. *Halldén-completeness in intermediate logics is undecidable (i.e. no algorithm exists which is capable of deciding given a formula A, whether or not logic $Int + A$ is Halldén-complete).*

Undecidable are also such properties of intermediate logics as finite model property, decidability (and some others; see [1]). These properties as well as the disjunction property are preserved under transferring from intermediate logic to its minimal and maximal companions via Gödel translation and vice versa. By means of this preservation theorem one can easily show the undecidability of the disjunction property, finite model property and decidability for normal logics containing $S4$ or even $S4Grz$.

Unfortunately the preservation theorem does not hold for Halldén-completness.

THEOREM 2. *There exist intermediate logics which are Halldén-complete but their normal modal companions are Halldén-incomplete.*

The intermediate logic determined by the frame shown in the figure below is one of the examples of such logics.



Thus the undecidability of Halldén-completeness in modal logics is not an immediate consequence of Theorem 1. Nevertheless, in the same manner as Theorem 1, we can prove

THEOREM 3. *Halldén-completeness in normal modal logics containing $S4$ (or even $S4Grz$) is undecidable.*

In order to prove this theorem we have obtained two sufficient conditions, coast in syntactic terms, for Halldén-completeness in normal modal logics containing $S4Grz$. The conditions are applied to canonical formulas [5] which can be used for axiomatization of every extensions of $S4$. As the consequences of the conditions we obtain the following results.

THEOREM 4. *There exists a continuum of modal logics which are Halldén-complete but do not have the disjunction property.*

THEOREM 5. *There exists a continuum of Halldén-complete modal logics having the disjunction property.*

In order to prove this theorem we used the construction from [3].
Moreover, there holds the following.

THEOREM 6. *There exists a continuum of Halldén-incomplete modal logics having the disjunction property.*

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