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COMMON LOGIC OF BINARY CONNECTIVES HAS FINITE MAXIMALITY DEGREE (Preliminary report)

Let F_p be a set of all *properly* binary Boolean functions $f : 2^2 \longrightarrow 2$, i.e., f depends on both arguments. Of the 16 binary truth functions 10 belongs to F_p, namely $\vee, \rightarrow, \leftrightarrow, \leftarrow$ (reverse implication), $\uparrow$ (Sheffer function) and the duals of these. For $F \subseteq F_p$ let $\vdash_F$ denote the common logic[1] of the $f \in F$ in the propositional language with *one* binary function symbol, $\vdash_F = \bigcap_{f \in F} \models_{2^f}$, where 2^f denotes the 2-element matrix $((2, f), 1)$. A study of $\vdash_F$ is useful for various purposes, e.g. for information processing, see [3]. $\vdash_F$ axiomatizes the common sequential rules of the $f \in F$. It needs not to have tautologies but this is a minor point. Particularly interesting is the question how ambiguous $\vdash_F$ actually is, i.e., how much information in form of additional rules needs a system of information proccesing dealing with $\vdash_F$ in order to identify a connective $f \in F$. This clearly amounts to an analysis of the strenghtenings $\vdash \supset \vdash_F$. Our main result is

Theorem 1. *$\vdash_{F_p}$ (and hence $\vdash_F$ for each $F \subseteq F_p$) has finitely many strenghtenings only. All these are determined by finitely many finite matrices.*

In other words, $\vdash_{F_p}$ has finite degree of maximality (see [4] for basic notions). $\vdash_{F_p}$ has a huge number of strenghtenings. Presently we only know that its number is less than 10^{45}. However, it has precisely 36 maximal

[1] A *logic* is here a structural consequence relation denoted by $\vdash$ or a similar symbol $\vdash$ is *non-trivial* if not $\alpha \vdash \beta$ for all formulas α, β. We omit the improper binary truth functions from our consideration because they are less interesting and cause some additional technical problems.

(nontrivial) strenghtenings, including the $\models_{2^f}$ for $f \in F_p$. The remaining 26 are 2^k-valued, $2 \leq k \leq 5$. For $|F| \leq 4$ the maximality degree of $\vdash_F$ is relatively small and can be computed by hand.

Theorem 1 easily follows from Theorem 2 and the Lemma below. $S\mathbf{K}$, $P\mathbf{K}$ denote the class of submatrices and of direct products of members of a class $\mathbf{K}$ of matrices, respectively. $\mathbf{t0}$ and $\mathbf{t1}$ denote the $\mathbf{1}$-element matrices whose element is designated and not designated, respectively. $\mathbf{K} \neq \emptyset$ implies $\mathbf{t1} \in P\mathbf{K}$ ($\mathbf{t1}$ appears as the power of some $\mathcal{A} \in \mathbf{K}$ with the empty index set). If $\mathbf{K}, \mathbf{M}$ are classes of matrices or single matrices we write $\mathbf{M} \equiv \mathbf{K}$ for $\models_{\mathbf{M}} = \models_{\mathbf{K}}$. Clearly $\mathbf{K} \cup \{\mathbf{t1}\} \equiv \mathbf{K}$ but $\mathbf{K} \cup \{\mathbf{t0}\} \equiv \mathbf{K}$ only if $\models_{\mathbf{K}}$ has no tautologies. A matrix $\mathcal{A}$ is *trivial* if either $\mathcal{A} \equiv \mathbf{t0}$ or $\mathcal{A} \equiv \mathbf{t1}$. Call $\mathbf{K}$ *closed* if for each nontrivial $\mathcal{A} \in SP\mathbf{K}$ there is some $\mathbf{M}_{\mathcal{A}} \subseteq \mathbf{K}$ with $\mathcal{A} \equiv \mathbf{M}_{\mathcal{A}}$. If $\vdash \, = \, \models_{\mathbf{K}}$ for some closed $\mathbf{K}$ then $\mathbf{K}$ is said to be a *closed semantics* for $\vdash$.

Lemma ([3]). *$\vdash$ has finite degree of maximality iff $\vdash$ has a closed semantics $\mathbf{M}, \mathbf{M}$ finite. If $|\mathbf{M}| = n$ then $\vdash$ has maximality degree $< 2^{n+1}$.*

The proof follows essentialy from a well-known result of [4] which implies that $\mathbf{K}$ is closed iff each $\vdash' \supseteq \models_{\mathbf{K}}$ has a representation $\vdash' = \models_{\mathbf{K}'}$ for some $\mathbf{K}' \subseteq \mathbf{K} \cup \{\mathbf{t0}\}$.

Let $\times\mathbf{M}$ denote the direct product of all members of a set $\mathbf{M}$ of matrices ($\times\emptyset = \mathbf{t1}$ and $\times\{\mathcal{A}\} = \mathcal{A}$). Put $P^*\mathbf{K} = \{\times\mathbf{M} : \mathbf{M} \subseteq \mathbf{K}\}$. Clearly, $|P^*\mathbf{K}| = 2^n$ provided $|\mathbf{K}| = n$.

Theorem 2. *For each $F \subseteq F_p, P^*\{2^f : f \in F\}$ is a closed semantics for $\vdash_F$.*

The proof of Theorem 2 which generalizes the results from [3] is essentially based on the fact that $\rightarrow, \leftrightarrow, \leftarrow, \uparrow$ are independent in the sense of [1] and that the variety $\mathbf{V}$ generated by the grupoids $(2, \rightarrow), (2, \leftrightarrow), (2, \leftarrow)$, $(2, \uparrow)$ is strongly irregular, i.e. there is a term $\sigma(x, y)$ such that in $\mathbf{V}$ holds the equation $\sigma(x, y) = x$ ([2, Example 1.7]).

The maximality degree of $\vdash_F$ strongly grows with $|F|$ but essentially depends also on the composition of F. E.g., for $|F| = 2$ it is ≤ 10 and this bound is realized for $\{\uparrow, \downarrow\}$ ($\downarrow$ the dual of $\uparrow$) as easily follows from Theorem 2. On the other hand in many cases of $F := \{f, g\}, \models_{2^f}$ and $\models_{2^g}$ are the only proper nontrivial strenghtenings of $\vdash_F$. An example is $F := \{\rightarrow, \leftrightarrow\}$. In this case $\{2^{\rightarrow}, 2^{\leftrightarrow}\}$ is already closed because $2^{\rightarrow} \times 2^{\leftrightarrow} \equiv \{2^{\rightarrow}, 2^{\leftrightarrow}\}$. Since each $\vdash\ \supseteq\ \vdash_F$ has tautologies, $\models_{2^{\rightarrow}}$ and $\models_{2^{\leftrightarrow}}$ are indeed the only proper nontrivial strenghtenings of $\vdash_F$. Call $F \subseteq F_P$ ($|F| \geq 2$) *nice* whenever $\{2^f : f \in F\}$ is already closed, as in the last example. For a nice F, the $f \in F$ have a maximum of common rules, or, the calculus $\vdash_F$ is ambiguous to minimal extend. In particular, the only maximal strenghtenings of $\vdash_F$ are the $\models_f$ for $f \in F$. From Theorem 2 it easily follows that F is nice if and only if F consists of some or all of the familiar connectives $\wedge, \vee, \rightarrow, \leftrightarrow$ and $\leftarrow$ which is essentially the same as $\rightarrow$). E.g., for $F_1 = \{\wedge, \vee, \rightarrow\}$, the favoured system of binary connectives, $\vdash_{F_1}$ has 7 proper nontrivial strenghtenings only. Consider $F_2 = \{\wedge, +, \rightarrow\}$, i.e. "or" is replaced by "either-or". $\vdash_{F_2}$ has nearly twice as many strenghtenings as has $\vdash_{F_1}$ which might explain to some extend the preference of F_1.

References

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