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INTERPRETATION OF RELEVANT LOGICS IN A LOGIC OF TERNARY RELATIONS

In Orłowska (1988) relational proof systems for various modal logics and for intuitionistic logic have been presented. The systems are based on interpretation of modal logics in the logic of binary relations. In the present paper the method is extended to capture the logics whose relational counterpart is a logic of ternary relations.

We give a semantic formulation of relevant logics investigated in Anderson and Belnap (1975). The Kripke-style semantics for the logics is due to Routley and Meyer (1973).

Formulas of the language of propositional relevant logics are constructed with symbols from the following disjoint sets:
$VARPROP$ – an infinite, denumerable set of propositional variables, and $\{\neg, \vee, \wedge, \rightarrow, \circ\}$ – the set of propositional operations of negation, disjunction conjunction, implication and intensional conjunction (or fusion), respectively.

The set of formulas is the smallest set including the set $VARPROP$ and closed with respect to the propositional operations. For formulas A and B their intensional conjunction is traditionally written AB instead of $A \circ B$.

Semantics of the language is defined in terms of the notions of model and satisfability of formulas in a model. By a model we mean a system

$$M = (U, O, *, R, m)$$

where U is a nonempty set, whose elements are called states or set-ups, $O \in U$ is a distinguished real state, $* : U \rightarrow U$ is a function in the set of states, $R \subseteq U \times U \times U$ is a ternary relation in the set U, and m :

$VARPROP \to P(U)$ is a meaning function which assigns sets of states to propositional variables. Moreover, it is assumed that in every model the following conditions are satisfied for any $a, b, c, t, u \in U$

$M1$ $ROaa$

$M2$ (i) $ROab$ and $Ratu$ imply $Rbtu$

(ii) $ROab$ and $a \in m(p)$ imply $b \in m(p)$ for any $p \in VARPROP$

$M3$ $Rabc$ implies Rac^*b^*

$M4$ $ROa^{**}a$.

We say that in a model M a state a satisfies a formula A ($M, a\ sat\ A$) iff the following conditions are satisfied:

$M, a\ sat\ p$ iff $a \in m(p)$ for $p \in VARPROP$

$M, a\ sat\ \neg A$ iff not $M, a^*\ sat\ A$

$M, a\ sat\ A \vee B$ iff $M, a\ sat\ A$ or $M, a\ sat\ B$

$M, a\ sat\ A \wedge B$ iff $M, a\ sat\ A$ and $M, a\ sat\ B$

$M, a\ sat\ A \to B$ iff for all $x, y \in U$ if $Raxy$ and $M, z\ sat\ A$ then $M, y\ sat\ B$

$M, a\ sat\ AB$ iff there are $x, y \in U$ such that $Rxya$ and $M, x\ sat\ A$, and $M, y\ sat\ B$.

A formula A is said to be true in a model M iff $M, O\ sat\ A$. A formula A is valid iff A is true in all models.

For various relevant logics their models satisfy some additional postulates. Below we give the list of those conditions. For $a, b, c, d \in U$ we denote by R^2abcd the fact that there is $x \in U$ such that $Rabx$ and $Rxcd$, and $R^2a(bc)d$ is to mean that there is an $x \in U$ such that $Rbcx$ and $Raxd$.

$M5$ R^2abcd implies $R^2a(bc)d$

$M6$ R^2abcd implies $R^2b(ac)d$

$M7$ R^2abcd implies R^2acbd

$M8$ $Rabc$ implies R^2abbc

$M9$ Raa^*a

$M10$ $Rabc$ implies $ROac$ or $RObc$

$M11$ $Rabc$ implies $ROac$

$M12$ $Rabc$ implies $R^2a(ab)c$.

Let OB be a nonempty set of objects, and let $* : OB \to OB$ be a function in OB. Consider a family of ternary relations in OB and let R be a distinguished relation. We define relational operations $\neg, \vee, \wedge, \to, \circ$ which, for the sake of simplicity, are denoted in the same way as their propositional counterparts in relevant logics. Let $A, B \subseteq OB^3$, then we put:

$\neg A = \{(a, b, c) : (a^*, b, c) \notin A\}$

$A \vee B = \{(a, b, c) : (a, b, c) \in A$ or $(a, b, c) \in B\}$

$A \wedge B = \{(a, b, c) : (a, b, c) \in A$ and $(a, b, c) \in B\}$

$A \to B = \{(a, b, c) :$ for every $x, y \in OB$ if $(a, x, y) \in R$ and $(x, b, c) \in A$ then $(y, b, c) \in B\}$

$AB = \{(a, b, c) :$ there are $x, y \in OB$ such that $(x, y, a) \in R, (x, b, c) \in A$ and $(y, b, c) \in B\}$.

As usually $-A = OB^3 - A$.

A relation $A \subseteq OB^3$ is said to be an ideal relation iff for any $x, y, z \in OB$, if $(x, y, z) \in A$, then for any $t, u \in OB$, $(x, t, u) \in A$. Hence a relation A is ideal if it is of the form $A = X \times OB \times OB$ for a certain $X \subseteq OB$.

Proposition 1. *The set of ideal relations in a set OB is closed with respect to the relational operations* $-, \neg, \vee, \wedge, \to, \circ$.

Expressions of the language of logic LT are constructed with symbols from the following pairwise disjoint sets:

$VAROB$ an infinite, denumerable set of object variables

CON an infinite, denumerable set of relational constants

$\{O, *\}$ the set consisting of an object constant O and unary object operation $*$

$\{R, \neg, -\}$ the set consisting of an relational constant R such that $R \notin CON$ and unary relational operations $\neg$ and $-$

$\{\vee, \wedge, \to, \circ\}$ the set of binary relational operations.

The set EOB of object expressions is the smallest set satisfying the following conditions:

$VAROB \subseteq EOB, O \in EOB$

$a \in EOB$ implies $a^* \in EOB$

The set $EREL$ of relational expressions is the smallest set satisfying the following conditions:

$CON \subseteq EREL$

If $A, B \in EREL$ then $\neg A, -A, A \wedge B, A \vee B, A \to B, AB \in EREL$.

Set FOR of formulas is the smallest set satisfying the conditions:

If $x, y, z \in EOB$ and $A \in EREL$, then $Axyz \in FOR$

$Rxyz, -Rxyz \in FOR$ for any $x, y, z \in EOB$.

A formula $Axyz$ is said to be nondegenerated iff $A \neq R$.

By a model of relational language we mean a system of the form

$$M = (OB, O, *, R, m)$$

such that $OB \neq \emptyset, O \in OB$ is an object which provides an interpretation of the constant O, $* : OB \to OB$ is a function in OB which provides an interpretation of the operation $*, R \subseteq OB \times OB \times OB$ is a relation which provides interpretation of the relational constant R, $m : EREL \to P(OB^3)$ is a meaning function which assigns ternary relations to relational expressions, and moreover the following conditions are satisfied:

$m0$ If $P \in CON$, then $m(P) = X \times OB \times OB$ for a certain $X \subseteq OB$

$m1$ $(O, a, a) \in R$ for any $a \in OB$

$m2$ (i) If $(O, a, b) \in R$ and $(a, t, u) \in R$, then $(b, t, u) \in R$
(ii) If $(O, a, b) \in R$ and $(a, t, u) \in m(P)$, then $(b, t, u) \in m(P)$ for every $P \in CON$

$m3$ $(a, b, c) \in R$ implies $(a, c^*, b^*) \in R$

$m4$ $(O, a^{**}, a) \in R$

$m5$ $m(-A) = -m(A)$

$m6$ $m(\neg A) = \neg m(A)$

$m7$ $m(A \vee B) = m(A) \vee m(B)$

$m8$ $m(A \wedge B) = m(A) \wedge m(B)$

$m9$ $m(A \to B) = m(A) \to m(B)$

$m10$ $m(AB) = m(A)m(B)$

By a valuation we mean a function $v : VAROB \to OB$ assigning objects to object variables. We say that in a model M a valuation v satisfies a relational formula $Axyz$ ($M, v\ sat\ Axyz$) iff the following condition is satisfied:

$M, v\ sat\ Axyz$ iff $(v(x), v(y), v(z)) \in m(A)$, where A is R or a relational expression.

Formula $Axyz$ is true in a model M iff M, $v\ sat\ Axyz$ for every valuation v. A formula is valid iff it is true in all models.

We define two kinds of deduction rules for the relational logic: Decomposition rules, which enable us to decompose relational formulas into some simpler formulas, depending on symbols of relational operations occurring in the formulas; Specific rules, corresponding to semantical postulates which are assumed in the models of the relational logic. The rules apply to finite sequences of formulas. As a result of application of a rule we obtain a single sequence, a pair of sequences or a triple of sequences of formulas. Let K and H denote finite (possibly empty) sequences of relational formulas, let A, B be relational expressions, and let a, b, c, t, u be object expressions. We admit the following rules:

Decomposition rules

$(--)$ $$\frac{K, --Aatu, K}{K, Aatu, H}$$

$(\neg)$ $$\frac{K, \neg Aatu, H}{K, -Aa^*tu, H}$$

$(-\neg)$ $$\frac{K, -\neg Aatu, H}{K, Aa^*tu, H}$$

$(\vee)$ $$\frac{K, (A \vee B)atu, H}{K, Aatu, Batu, H}$$

$(-\vee)$ $$\frac{K, -(A \vee B)atu, H}{K, -Aatu, H \qquad K, -Batu, H}$$

$(\wedge)$ $$\frac{K, (A \wedge B)atu, H}{K, Aatu, H \qquad K, Batu, H}$$

$(-\wedge)$ $$\frac{K, -(A \wedge B)atu, H}{K, -Aatu, -Batu, H}$$

$(\to)$ $$\frac{K, (A \to B)atu, H}{K, -Raxy, -Axtu, Bytu, H}$$

where x, y are object variables which do not occur in any formula above the line

$$(-\rightarrow) \qquad \frac{K, -(A \rightarrow B)atu, H}{H_1 \qquad H_2 \qquad H_3}$$

where
$H_1 = K, Raxy, -(A \rightarrow B)atu, H$
$H_2 = K, Axtu, -(A \rightarrow B)atu, H$
$H_3 = K, -Bytu, -(A \rightarrow B)atu, H$
x, y are arbitrary object expressions

$$(\circ) \qquad \frac{K, (AB)atu, H}{H_1 \qquad H_2 \qquad H_3}$$

where
$H_1 = K, Axtu, (AB)atu, H$
$H_2 = K, Bytu, (AB)atu, H$
$H_3 = K, Rxya, (AB)atu, H$
x, y are arbitrary object expressions

$$(-\circ) \qquad \frac{K, -(AB)atu, H}{K, -Rxya, -Axtu, -Bytu, H}$$

where x, y are object variables which do not occur in any formula above the line

Specific rule

$$(R1) \qquad \frac{K, Patu, H}{K, Pazw, H}$$

where z, w are arbitrary object expressions, and P is a relational constant

In what follows rule (Ri) corresponds to condition (Mi).

$$(R2) \qquad \frac{K, Aatu, H}{K, ROaz, Aatu, H \qquad K, Aztu, Aatu, H}$$

where z is an arbitrary object expression, A denotes R or a relational expression.

$$(R3) \qquad \frac{K, Rac^*b^*, H}{K, Rabc, Rac^*b^*, H}$$

$$(R5) \qquad \frac{K, -Rabx, -Rxcd, H}{K, -Rbcz, -Razd, -Rabx, -Rxcd, H}$$

(*R*6) $$\frac{K, -Rabx, -Rxcd, H}{K, -Racz, -Rbzd, -Rabx, -Rxcd, H}$$

(*R*7) $$\frac{K, -Rabx, -Rxcd, H}{K, -Racz, -Rzbd, -Rabx, -Rxcd, H}$$

(*R*8) $$\frac{K, -Rabc, H}{K, -Rabz, -Rzbc, -Rabc, H}$$

In $(R5)$ – $(R8)$ z is an object variable which does not occur in any formula above the line

(*R*10) $$\frac{K, -Rabc, H}{K, -ROac, -Rabc, H \qquad K, -RObc, -Rabc, H}$$

(*R*11) $$\frac{K, ROac, H}{K, Razc, ROac, H}$$

where z is an object variable

(*R*12) $$\frac{K, -Rabc, H}{K, -Rabz, -Razc, -Rabc, H}$$

where z is an object variable which does not occur in any formula above the line

A sequence of formulas is said to be fundamental iff it contains formulas of one of the following forms for any object expressions a, t, u:
$(f1)$ $Aatu, -Aatu$ where A is R or a relational expression
$(f2)$ $ROaa$
$(f3)$ $ROa^{**}a$
$(f4)$ $Raa^{*}a$

Fundamental sequences play the role of axioms in the deduction system. Sequences $(f2)$, $(f3)$, $(f4)$ correspond to conditions $(M1)$, $(M4)$, $(M9)$, respectively. A sequence K of formulas is valid iff for every model M and every valuation v there is a formula F in K such that M, v *sat* F.

Proposition 2. *(a) Fundamental sequences are valid.*
(b) In every deduction rule the sequences above the line is valid iff all the sequences under the line are valid.

Given a formula $Aatu$, where A is a compound relational expression, we can decompose it by successive application of the decomposition rules, or we can apply a specific rule. In the process of application of the rules we form a tree whose vertices consist of finite sequences of formulas. Each vertex has at most three successors. We stop applying formulas in a vertex after obtaining a fundamental sequences. Trees obtained in this way are called decomposition trees.

A branch of a decomposition tree is said to be fundamental if it contains a vertex with a fundamental sequence. A decomposition tree is fundamental if all of its branches are fundamental. Observe that every fundamental decomposition tree is finite.

PROPOSITION 3. *The following conditions are equivalent:*
(a) A nondegenerate formula F of the relational logic is valid
(b) There is a fundamental decomposition tree of F.

We define the translation function t from formulas of relevant logic into relational formulas. Assume that we are given a one-to-one mapping $t' : VARPROP \rightarrow CON$ assigning relational constants to propositional variables. Next, we define:

$t(p) = t'(p)$ for $p \in VARPROP$
$t(\neg A) = \neg t(A),\ t(A \vee B) = t(A) \vee t(B),\ t(A \wedge B) = t(A) \wedge t(B),$
$t(A \rightarrow B) = t(A) \rightarrow t(B),\ t(AB) = t(A)t(B).$

PROPOSITION 4. *The following conditions are equivalent:*
(a) A formula F of relevant logic is valid
(b) Relational formula $t(F)Otu$ is valid in the relational logic.

EXAMPLE. We show a relational proof of the formula $A \rightarrow ((A \rightarrow B) \rightarrow B)$.

$$A \rightarrow ((A \rightarrow B) \rightarrow B)Otu$$

| $(\rightarrow)$
| new expressions: $= x, y$

$$-ROxy, -Axtu, ((A \rightarrow B) \rightarrow B)ytu$$

| $(\rightarrow)$
| new expressions: $= v, w$

$$-ROxy, -Axtu, -Ryvw, -(A \to B)vtu, Bwtu$$

$(- \to)$
new expressions: $= x, w$

$-ROxy, -Axtu,$ $-Ryvw, Bwtu,$ $Rvzw,$ $-(A \to B)vtu,$	$-ROxy, -Axtu,$ $-Ryvw, Bwtu,$ $Axtu,$ $-(A \to B)vtu,$ fundamental	$-ROxy, -Axtu,$ $-Ryvw, Bwtu,$ $-Bwtu,$ $-(A \to B)vtu,$ fundamental

$(R7)$ to $-ROxy, -Ryvw$
new expression: $= z$

$$-ROxy, -Axtu, -Ryvw, Bwtu, Rvxw, -(A \to B)vtu, -ROvz, -Rzxw$$

$(R2)$ to $Rvxw$
new expression: $= z$

$-ROxy, -Axtu, -Ryvw,$ $Bwtu, Rvxw, -(A \to B)vtu,$ $ROvz, -Rxzw, ROvz,$ fundamental	$-ROxy, -Axtu, -Ryvw,$ $Bwtu, Rvxw, -(A \to B)vtu,$ $-ROuz, -Rzxw, Rzxw,$ fundamental

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