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A NOTE ON SOME PROPERTY OF PURELY IMPLICATIONAL PROPOSITIONAL CALCULI

Let F be the set of formulas built from an infinite set $Var = \{p_1, p_2, \ldots\}$ of propositional variables and the binary connective $\rightarrow$.

In this paper we consider those sets $X \subseteq F$ for which the following condition holds:

(*) $$A \rightarrow B \in X \Leftrightarrow \forall_{e \in End(F)}[e(A) \in X \Rightarrow e(B) \in X]$$

for every pair $A, B \in F$.

$End(F)$ is the set of all endomorphisms of the algebra $(F, \rightarrow)$.

Lemma 1. *If $X \subseteq F$ satisfies the condition (*) then for any $A, B, C \in F$:*

(i) $$A \rightarrow B \in X \Rightarrow \forall_{e \in End(F)}\ e(A \rightarrow B) \in X,$$

(ii) $$A \rightarrow B, A \in X \Rightarrow B \in X,$$

(iii) $$A \rightarrow A \in X,$$

(iv) $$(A \rightarrow B) \rightarrow (C \rightarrow (A \rightarrow B)) \in X,$$

(v) $$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \in X,$$

(vi) $$((A \rightarrow (B \rightarrow A)) \rightarrow A) \rightarrow A \in X,$$

Lemma 2. *For every $p, q \in Var$, $p \neq q$, and for any $X \subseteq F$ which has the property $(*)$ the following conditions are equivalent:*

(c1) $$p \to (q \to p) \in X,$$

(c2) $$X = Sb(X),$$

where $Sb(X) = \{B \in F : \exists_{A \in X} \exists_{e \in End(F)}\ B = e(A)\}$.

Now let H be the set of all purely implicational theses of the intuitionistic propositional calculus. Then we have (cf. [1])

Lemma 3. *If $H \subseteq X = Sb(X)$ then X satisfies the condition $(*)$ for every $X \subseteq F$.*

Let P be the family of all sets $X \subseteq F$ which have the property $(*)$ and $X = Sb(X)$. Then from lemmas 1, 2 and 3 we obtain

Theorem. *The set H is the smallest set in the family P.*

Conjecture. *There exists a set $X \subseteq F$ such that X is not closed with respect to the substitution rule and X satisfies the condition $(*)$.*

References

[1] T. Prucnal, *On the structural completeness of some pure implicational calculi*, **Studia Logica** vol. 30 (1972), pp. 45–52.

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