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ŁUKASIEWICZ LOGIC AND WAJSBERG ALGEBRAS[1]

The object of this paper is to prove that the class of Wajsberg algebras is the Equivalent Variety Semantics for Łukasiewicz Logic (in the sense of W. Blok and D. Pigozzi [BP]).

We recall that (infinite-valued) *Łukasiewicz Logic* $\mathcal{L}_\infty$ is the logic defined over the free algebra of type (2,1) with axioms:

$$\phi \to (\psi \to \phi)$$

$$(\phi \to \psi) \to ((\psi \to \eta) \to (\phi \to \eta))$$

$$((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi))$$

$$(\neg\phi \to \neg\psi) \to (\psi \to \phi)$$

and with *Modus Ponens* as a primitive rule.

Now we are going to define Wajsberg algebras and in Corollary of Theorem 3 we will obtain that the class of all Wajsberg algebras is the Equivalent Variety Semantics for the logic $\mathcal{L}_\infty$. Let $\mathbf{W} = (W, \to, \neg)$ be an algebra of type (2,1), we say that $\mathbf{W}$ is a *Wajsberg algebra* when it satisfies the following equations:

W.1. $$(x \to x) \to y = y$$

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$W.2.$ $$(x \to y) \to ((y \to z) \to (x \to z)) = x \to x$$

$W.3.$ $$((x \to y) \to y = (y \to x) \to x$$

$W.4.$ $$(\neg x \to \neg y) \to (y \to x) = x \to x$$

Wajsberg algebras, just defined, are polynomialy equivalent to Wajsberg algebras defined in [R] and to Chang's MV-algebras defined in [Ch] (see [FRT]).

To show main result of this paper we will use Theorem 5.1 of [BP]:

Theorem 1. *Let $\mathcal{L}$ be a logic and K a quasivariety. The following are equivalent:*

(i) $\mathcal{L}$ is algebraizable with Equivalent Semantics K.

(ii) For every algebra $\mathcal{A}$ the Leibniz operator $\Omega_{\mathcal{A}}$ is an isomorphism between lattices of $\mathcal{L}$-filters and of K-congruences of $\mathcal{A}$.

Let $\mathcal{A} = (A, \to, \neg)$ be an algebra of type (2,1). A subset D of A is called a *Wajsberg deductive system* (or $\mathcal{L}_\infty$-filter in the sense of [BP]), when it satisfies: for any $a, b, c \in A$

$WD.1$ $a \to (b \to a) \in D$
$WD.2.$ $(a \to b) \to ((b \to c) \to (a \to c)) \in D$
$WD.3.$ $((a \to b) \to b) \to ((b \to a) \to a) \in D$
$WD.4.$ $(\neg a \to \neg b) \to (b \to a) \in D$
$MP.$ $a, a \to b \in D$ implies $b \in D$ (*Modus Ponens*).

It is obvious that any subset of A containing a Wajsberg deductive system and closed under *Modus Ponens* is also a Wajsberg deductive system.

Lemma 2. *Let $\mathcal{A}$ be an algebra of type $(2,1)$ and $D \subset A$ a Wajsberg deductive system. Then for any $a, b, c \in A$, and any $d \in D$ we have the following:*

$WD.5.$ $a \to a \in D$
$WD.6.$ $a \to (b \to c) \in D$ *implies* $b \to (a \to c) \in D$
$WD.7.$ $(a \to b) \to ((c \to a) \to (c \to b)) \in D$
$WD.8.$ $(d \to a) \to a \in D$
$WD.9.$ $a \to (\neg a \to \neg d) \in D$ *and* $(a \to \neg d) \to \neg a \in D$
$WD.10.$ $a \to b \in D$ *implies* $\neg b \to \neg a \in D.$

PROOF. **WD.5.** If $d \in D$ then by $WD.1$ and MP we have $a \to d \in D$ for any $a \in A$. By $WD.2$ we have $(a \to (d \to a)) \to (((d \to a) \to a) \to (a \to a)) \in D$ so by MP, $((d \to a) \to a) \to (a \to a) \in D$. By $WD.2$, $WD.3$ and MP, we have $((a \to d) \to d) \to (a \to a) \in D$. Since $(a \to d) \to d \in D$ we have, by MP, $a \to a \in D$.

WD.6. If $a \to (b \to c) \in D$, then by $WD.2$ and MP, $((b \to c) \to c) \to (a \to c) \in D$ but by $WD.1$, $b \to ((c \to b) \to b) \in D$, so by $WD.2$ and MP, $b \to (a \to c) \in D$.

WD.7 It is an immediate consequence of $WD.2$ and $WD.6$.

WD.8 By $WD.1$ we have that $(a \to d) \to d \in D$, so by $WD.3$ and MP we have $(d \to a) \to a \in D$.

WD.9 By $WD.1$ we have $\neg a \to (\neg\neg d \to \neg a) \in D$ so by $WD.2$, $WD.4$ and MP we have $\neg a \to (a \to \neg d)$, and by $WD.6$, $a \to (\neg a \to \neg d) \in D$. On the other hand, by $WD.4$, $(\neg a \to \neg d) \to (d \to a) \in D$. Now using $WD.6$, and MP we have $(\neg a \to \neg d) \to a \in D$, so by $WD.2$ and MP: $(a \to \neg d) \to ((\neg a \to \neg d) \to \neg d) \in D$. By the first part of $WD.9$, $\neg a \to (\neg\neg a \to \neg d) \in D$, so by $WD.6$ and MP, $\neg\neg a \to (\neg a \to \neg d) \in D$ hence $((\neg a \to \neg d) \to \neg d) \to (\neg\neg a \to \neg d) \in D$. By $WD.4$ $(\neg\neg a \to \neg d) \to (d \to \neg a) \in D$, so using $WD.2$ and MP twice we have $(a \to \neg d) \to (d \to \neg a) \in D$, and by $WD.6$, $WD.2$ and MP, $(a \to \neg d) \to \neg a \in D$.

WD.10 Suppose $a \to b \in D$. By $WD.9$, $b \to (\neg b \to \neg d) \in D$, so by $WD.2$ and MP, $a \to (\neg b \to \neg d) \in D$, and by $WD.6$ we have $\neg b \to (a \to \neg d) \in D$. Hence by $WD.9$, MP and $WD.2$ we obtain $\neg b \to \neg a \in D$. □

Given an algebra $\mathcal{A}$ of type (2,1) we denote by $\mathcal{D}_W(\mathcal{A})$ the family of all Wajsberg Deductive System of $\mathcal{A}$, and by $Con_W(\mathcal{A})$ the family of all Wajsberg congruence relations of $\mathcal{A}$ (i.e. the congruence relations of $\mathcal{A}$ such that the quotient is a Wajsberg algebra). Then we have

THEOREM 3. *Let* $\mathcal{A} = (A, \to, \neg)$ *be an algebra of type* $(2,1)$*. The map*

$$\Theta : \mathcal{D}_W(\mathcal{A}) \longrightarrow Con_W(\mathcal{A}) : D \mapsto \theta_D = \{(a,b) \in A^2 : a \to b, b \to a \in D\}$$

is an order isomorphism, whose inverse is:

$$\mathcal{D} : Con_W(\mathcal{A}) \longrightarrow \mathcal{D}_W(\mathcal{A}) : \theta \mapsto D_\theta = \{a \in A : (a, a \to a) \in \theta\}.$$

PROOF. **I**) Let D be a Wajsberg Deductive System of $\mathcal{A}$. Using $WD.5$, $WD.2$ and MP we obtain that θ_D is an equivalence relation on A. Moreover using $WD.2$ and MP we have that θ_D satisfies the substitution property relative to $\to$. Using $WD.10$ we have that θ_D satisfies the substitution property relative to $\neg$. So $\theta_D \in Con_W(\mathcal{A})$. On the other hand if $a, b \in D$ them, by MP, $(a,b) \in \theta_D$. If $a \in D$ and $b \notin D$, then by MP, $a \to b \notin D$, so $(a,b) \notin \theta_D$. That is D is an equivalence class and $(a \to a)/\theta_D = D$. From $WD.8$, $WD.1 - WD.4$, we deduce that $\mathcal{A}/\theta_D$ is a Wajsberg algebra. Thus $\theta_D \in Con_W(\mathcal{A})$. It is clear that if D and D' are Wajsberg Deductive Systems and $D \subset D'$, then $\theta_D \subset \theta_{D'}$.
II) Let $\theta \in Con_W(\mathcal{A})$, then using the properties of Wajsberg algebras (see [FRT]) we have that D_θ is the greatest element of $\mathcal{A}/\theta$ and $\{D_\theta\}$ is a Wajsberg Deductive System of $\mathcal{A}/\theta$. Then it is easy to see that D_θ is a Wajsberg Deductive System of $\mathcal{A}$. Moreover, if $\theta, \theta' \in Con_W(\mathcal{A})$, then $\theta \subset \theta'$ implies $D_\theta \subset D_{\theta'}$. Moreover, $(a,b) \in \theta_{D_\theta}$ iff $a \to b, b \to a \in D_\theta$ iff $a/\theta \to b/\theta = D_\theta$, $b/\theta \to a/\theta = D_\theta$ iff $a/\theta = b/\theta$ iff $(a,b) \in \theta$, so $\theta_{D_\theta} = \theta$. On the other hand it is easy to see that $D_{\theta_D} = D$. □

COROLLARY. *For any algebra* $\mathcal{A} = (A, \to, \neg)$ *of type* $(2,1)$*, the Leibniz operator* $\Omega_\mathcal{A}$ *is an isomorphism from* $\mathcal{D}_W(\mathcal{A})$ *onto* $Con_W(\mathcal{A})$*. Hence the Lukasiewicz logic is algebraizable with the class of all Wajsberg algebras as its equivalent variety semantics.*

PROOF. For any $D \in \mathcal{D}_W(\mathcal{A})$, θ_D is defined elementarily over the matrix $(\mathcal{A}, D)$ without equality, is a congruence on $\mathcal{A}$ and is compatible with D, so by Theorem 1.6 of [BP], $\theta_D = \Omega_\mathcal{A}(D)$. Thus by Theorem 1 and 3 we have that $\mathcal{L}_\infty$ is algebraizable with Equivalent Semantics the Variety of Wajsberg Algebras. □.

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