Bulletin of the **S**ection of **L**ogic
Volume 19/2 (1990), pp. 56–57

reedition 2005 [original edition, pp. 56–57]

Marek Pałasiński

THE ANSWER TO DZIOBIAK'S QUESTION

The following question was posed by W. Dziobiak [2]:

Is every quasivariety with the relative congruence extension property a variety?

The answer to the question is negative: the quasivariety of BCK-algebras has the relative congruence extension property (a simple proof is given below) and is not a variety (see [5]). Moreover the lattice of relative congruences of any BCK-algebra is distributive (see [4]).

The notation and terminology are standard. For basic universal algebra facts the reader is referred to [1] and for BCK-algebra background to [3].

Let us start with necessary definitions.

Let Q denote a quasivariety and $\mathcal{A} \in Q$. We define the set of relative congruences of $\mathcal{A}$, $Con_Q\mathcal{A} := \{\Theta \in Con\mathcal{A} : A/\Theta \in Q\}$. We say that Q has the relative congruence extension property if for any $\mathcal{A}, \mathcal{B} \in Q$ such that $\mathcal{A}$ is subalgebra of $\mathcal{B}$ and any $\Theta \in Con_Q\mathcal{A}$ there is $\Psi \in Con_Q\mathcal{B}$ such that $\Theta = \Psi \cap A^2$.

Let in the sequel Q denotes the quasivariety of all BCK-algebras.

Let $\mathcal{A} = (A, *, 0)$ be a BCK-algebra. A subset I of the set A is called an ideal of $\mathcal{A}$ if

1. $0 \in I$;
2. for any $x, y \in A$ if $x \in I$ and $y * x \in I$ then $y \in I$.

Theorem 1. (Iseki, Tanaka [3]). *If $\mathcal{A} = (A, *, 0)$ is a BCK-algebra, $B \subseteq A$ then there is the smallest ideal I of $\mathcal{A}$ containing B (the ideal generated by B) and*

$$I = \{a \in A : \exists_{n \in N} \exists_{b_1,\ldots,b_n \in B} (\ldots(a * b_1) * \ldots) * b_n = 0\}.$$

Corollary. *For any BCK-algebras $\mathcal{A}, \mathcal{B}$ such that $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and any ideal J of $\mathcal{B}$ there is an ideal I of $\mathcal{A}$ such that $I \cap B = J$.*

Proof. Take I the ideal of $\mathcal{A}$ generated by J.

The following was shown by Yutani [6].

Theorem 2. *For any BCK-algebra $\mathcal{A} = (A, *, 0)$ and $\Theta \in Con\mathcal{A}$ the following conditions are equivalent:*

(i) $\Theta \in Con_Q\mathcal{A}$

(ii) There is an ideal I of $\mathcal{A}$ such that for all $a, b \in A, a \equiv b(\Theta)$ if and only if $a * b, b * a \in I$.

Now it follows from Theorem 2 and Corollary

Fact. *The quasivariety of all BCK-algebras has the relative congruence extension property.*

References

[1] S. Burris and H. P. Shankapanavar, **A Course in Universal Algebra**, Springer-Verlag, New York, 1981.

[2] J. Czelakowski, personal communication.

[3] K. Iseki and S. Tanaka, *An introduction to the theory of BCK-algebras*, **Mathematica Japonica**, vol. 23 (1978), pp. 1–26.

[4] M. Pałasiński, *On ideal and congruence lattices of BCK-algebras*, **Mathematica Japonica**, vol. 26 (1981), pp. 543–544.

[5] A. Wroński, *BCK-algebras do not form a variety*, **Mathematica Japonica**, vol. 28 (1983), pp. 211-213.

[6] H. Yutani, *Quasi-commutative BCK-algebras and congruence relations*, **Mathematics Seminar Notes**, vol. 5 (1977), pp. 469–480.

Computer Science Departament
Jagiellonian University
Cracov, Poland

Department of Mathematics
Iowa State University
Ames, Iowa USA