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LOGICS OF RELATIONAL SYSTEMS

This is an abstract of paper which will be published in Studia Logica.

1. In this paper we consider pure first-order predicate logics (without function letters and individual constants). The symbols $x_1, x_2, \ldots$ are individual variables. The symbols $P^k_n(k, n = 1, 2, \ldots)$ are n-ary predicate letters. By an atomic formula we mean any formula of the form $P^k_n(x_{i_1}, \ldots, x_{i_n})$. The set of all atomic formulas is denoted by At. The set S of all formulas is the smallest set satisfying the conditions: $P^k_n(x_{i_1}, \ldots, x_{i_n}) \in S$, if $\alpha, \beta \in S$ then $\alpha \to \beta, \neg\alpha, \bigwedge_{x_k} \alpha \in S$.

Let $\mathcal{E}$ be the class of functions of substitution for atomic formulae (for details see [1], Def. 1.2, p. 216). Any function $e \in \mathcal{E}$ can be extended to an endomorphism $h^e : S \to S$.

The set of all theorems of the classical first-order predicate calculus will be denoted by L_2. Note that if $\alpha \in L_2$ then $h^e(\alpha) \in L_2$, for every $e \in \mathcal{E}$.

2. Let M be a non-empty set, $R_1, \ldots, R_n$ k_i-place relations over M; $f_1, \ldots, f_s$ functions respectively t_i-argument on M, where $t_i \geq 0$. Then the sequence $< M, R_1, \ldots, R_n, f_1, \ldots, f_k >$ is called a relational system and denoted by $\mathcal{M}$. A relational system $\mathcal{M}$ is finite if M is a finite set. The class of all finite relational systems will be denoted by Fin.

Let $\mathcal{M}$ be a relational system. Then by $J_{\mathcal{M}}$ we denote the first-order predicate language associated with $\mathcal{M}$. The set of all formulae of the language $J_{\mathcal{M}}$ is denoted by $F_{\mathcal{M}}$.

$E(\mathcal{M})$ is the set of all formulae of the language $J_{\mathcal{M}}$ which are true in $\mathcal{M}$.

Let $\Phi, \Psi \in F_{\mathcal{M}}$. Φ and Ψ are similar, in symbols $\Phi \approx \Psi$, if one of them can be obtained by changing some bound variables in the second one (for details see [1]). By $[...]_n$ we mean the operation which increases the indices of bound variables:

$$[\bigwedge_{x_k} \Phi]_n = \bigwedge_{x_{k+n+s}} \Phi(x_k/x_{k+n+s})]_{n_s}$$

where s is the greatest index among the indices of all individual variables occurring in Φ, (for the remaining logical constants and for the atomic formulae of $J_{\mathcal{M}}$ this operation is invariant). $\Phi(x_k/x_n)$ is the correct result of substituting x_n for any free occurrence of x_k in Φ, (cf. [1]).

For every relational system $\mathcal{M}$ we define the set $V_{\mathcal{M}}$ of functions as follows:

$$v \in V_{\mathcal{M}} \Leftrightarrow \quad \begin{array}{ll} \text{a.} & v : At \to F_{\mathcal{M}} \\ \text{b.} & Vf(v(\alpha)) = Vf(\alpha) \\ \text{c.} & v(\alpha(x_k/x_n)) \approx [v(\alpha)]_n(x_k/x_n), \end{array}$$

for any $\alpha \in At \subseteq S$ and for every $k, n \in N$. The symbol $Vf(\Phi)$ denotes the set of all free variables occurring in Φ. Every function $v \in V_{\mathcal{M}}$ can be extended to a homomorphism $h^v : S \to F_{\mathcal{M}}$.

A formula $\alpha \in S$ is said to be a predicate tautology of a relational system $\mathcal{M}$ if for every $v \in V_{\mathcal{M}} : h^v(\alpha) \in E(\mathcal{M})$. The set of all predicate tautologies of a relational system $\mathcal{M}$ will be denoted by $L(\mathcal{M})$ and called the predicate logic of the relational system $\mathcal{M}$.

Corollary 1. *For every relational system $\mathcal{M}$:*
(i) $L_2 \subseteq L(\mathcal{M})$,
(ii) $\alpha, \alpha \to \beta \in L(\mathcal{M}) \Rightarrow \beta \in L(\mathcal{M})$,
(iii) $\alpha \in L(\mathcal{M}) \Rightarrow \bigwedge_{x_k} \alpha \in L(\mathcal{M})$,
(iv) $\alpha \in L(\mathcal{M}) \Rightarrow \forall_{e \in \mathcal{E}}\ h^e(\alpha) \in L(\mathcal{M})$.

Corollary 2. *For every $\alpha \in S$:*

$$\alpha \in L_2 \Leftrightarrow \forall_{\mathcal{M}}\ \alpha \in L(\mathcal{M}).$$

Let $x_1, x_2, \ldots, x_n \notin Var(\alpha)$, where $Var(\alpha)$ is the set of all individual variables occurring in α. We define now the operation $K^n : S \to S$ as follows:

(a) $K^n(\alpha) = \alpha$, for $\alpha \in At$

(b) $K^n(\neg\alpha) = \neg K^n(\alpha)$
(c) $K^n(\alpha \to \beta) = K^n(\alpha) \to K^n(\beta)$
(d) $K^n(\bigwedge_{x_l} \alpha) = K^n(\alpha(x_1)) \wedge K^n(\alpha(x_2)) \wedge \ldots \wedge K^n(\alpha(x_n))$,

where $\alpha(x_i) = \alpha(x_l/x_i)$ and $\beta \wedge \gamma = \neg(\beta \to \neg\gamma)$.

Theorem 1. *For every closed formula $\alpha \in S$:*

$$\alpha \in L^2 \Rightarrow \forall_n \ K^n(\alpha) \in L^2.$$

Theorem 2. *For every closed formula $\alpha \in S$:*

$$\alpha \in L^{fin} \Leftrightarrow \forall_n \ K^n(\alpha) \in L^{fin},$$

where $L^{fin} = \bigcap\{L(\mathcal{M}) : \mathcal{M} \in Fin\}$.

References

[1] W. A. Pogorzelski and T. Prucnal, *Structural completeness of the first order predicate calculus*, **Bulletin de l'Academie Polonaise des Sciences, Serie des Sciences Mathematiques, Astronomiques et Phisiques**, vol. 22, no. 3 (1974).

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