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AN INTEGER-VALUED MATRIX CHARACTERISTIC FOR IMPLICATIONAL $S5$

The strict-implicational fragment $C5$ of $S5$ was first axiomatized in [1], C. A. Meredith having shown that an adequate base is provided by taking *modus ponens* ("MP") as sole rule together with the substitution instances of $CqCpp$, $CCpqCCqrCpr$ and, with 'P' short for 'Cpr', $CCCPqPP$.

The standard matrix characteristic for $C5$ may be extracted from the Henle matrix H given for full $S5$ in [2]. Our values is the subsets of the set Z_+ of positive integers. The designated value is the empty set, $\emptyset$, and a conditional $C\alpha\beta$ has the value $\emptyset$ if the value of its antecedent includes as a subset the value assigned to its consequent. I note that this matrix is obtained from the Henle matrix as described for full $S5$ in [2] by extracting its "strict-implicational part", that is the corresponding matrix for $LC\alpha\beta$; I have also dualized, interchanging sets and their complements (while replacing '$\subseteq$' with '$\supseteq$' throughout).

Of course this matrix employs a nondenumerably infinite set of values. The author knows of two other matrices characteristic for $C5$ described, via their full $S5$ counterparts, in the literature: one, due to Prior [4], employs as values the (nondenumerable) set of all denumerable sequences of 0's and 1's; the other, due to Rescher [5], uses the real numbers between 0 and 1 inclusive, again a nondenumerable set.

Now it has become a commonplace, since Lindenbaum first proved it in the 1920's that every sentential calculus has characteristic matrix that is, at worst denumerable. It can be a useful aid to thinking about such systems to discover single, denumerable matrices characteristic for them. The aim of this note is to describe such a matrix for $C5$, and to prove soundness and completeness.

1 The Multiples Matrix

The Multiples Matrix, **MM**, has as its set of values the set **N** of natural numbers: 1, 2, 3, ... together with 0. The sole designated value is 1. And the function ς associated with the connective C is defined for a and b in **N** so that $\varsigma(a, b) = 1$ if a is a multiple of b and $\varsigma(a, b) = 0$ otherwise. Thus, ς is the characteristic function of the relation *is a multiple of*.

SOUNDNESS THEOREM. *Each theorem of* $C5$ *is a tautology of* **MM**.

PROOF. It is clear that the rule MP preserves tautologousness: if α and $C\alpha\beta$ both take the value 1 for some assignment of values of **MM** to the letters occurring in them, then the value taken by α on that assignment must be 1, and must also be a multiple of the value taken by β, whence β's value must itself be 1. So it suffices to show that each of $C5$'s axioms is an **MM**-tautology.

Re. $CpCqq$: Whatever value is assigned to q is among its own multiples, so that Cqq takes the value 1. Whatever value is assigned to p is a multiple of 1. So for each assignment of values of **MM** to p and q, $CpCqq$ takes the value 1.

Re. $CCpqCCqrCpr$: Suppose there is an assignment of values – say a to p, b to q and c to r – according to which $CCpqCCqrCpr$ takes the value 0 in **MM**. Then Cpq must take the value 1 and $CCqrCpr$ the value 0. The first of these alleged facts assures that a is a multiple of b, and the second that $\varsigma(b, c) = 1$ while $\varsigma(a, c) = 0$. But then b is the multiple of c whence a, being a multiple of b must also be a multiple of c, contradicting the conclusion that $\varsigma(a, c) = 0$.

Re. $CCCPqPP$: Suppose that, for some assignment of values to letters, $CCPqP$ takes the value 1 and P the value 0. Then the value taken by CPq must be a multiple of that taken by P, that is, of 0. This can only be the case if CPq itself takes the value 0, whence the value taken by P must not be a multiple of the value of q. But the value of P is 0 and is thus a multiple of the value of q whatever the latter may be, since $0 = b \times 0$ for every natural number b.

2 Completeness

To show efficiently that each **MM**-tautology is a theorem of $C5$, it is convenient to employ a sequence of finite matrices defined along Henle lines:

DEFINITION. For each positive integer n, the matrix $\mathbf{H}_n$ is to have as values the subsets of $\{1, \ldots, n\}$, and as sole designated value the empty set, $\emptyset$. For values A and B of $\mathbf{H}_n$, $\varsigma(A, B)$ is to be $\emptyset$ if $A \supseteq B$ and $\{1, \ldots, n\}$ otherwise.

These matrices are dualized from some introduced by Scroggs [6], again interchanging sets and their complements while replacing $\subseteq$ with $\supseteq$. Accordingly, the following result may be recovered from the corresponding result for full $S5$ in [6], which I remark was apparently first established by Parry [3]:

LEMMA. *A wff of $C5$ in which at most n distinct sentence letters occur is a theorem of $C5$ iff it is a tautology of $\mathbf{H}_n$.*

Now it is easy enough to find, for each such $\mathbf{H}_n$, a submatrix $\mathbf{S}_n$ of the Multiples Matrix isomorphic to it. As values, $\mathbf{H}_n$ uses the subsets of $\{1, \ldots, n\}$. For the values of $\mathbf{S}_n$ we use 0, 1, the first n prime numbers, $P_1, \ldots, P_n$, and the square-free products of these primes.

LEMMA. *For $n \geq 2$, $\mathbf{H}_n$ is isomorphic to the submatrix $\mathbf{S}_n$ of* **MM**.

PROOF. Define the function h from the set of values of $\mathbf{H}_n$ to the set of values of $\mathbf{S}_n$ as follows. Let $h(\{i\}) = P_i$ for each positive integer i; let $h(\{1, \ldots, n\}) = 0$, with $h(\{i_1, \ldots, i_m\}) = P_{i_1} \times \ldots \times P_{i_m}$ otherwise; and let $h(\emptyset) = 1$. Thus h sends the empty subset of $\{1, \ldots, n\}$ to 1, singletons to distinct primes, pair-sets to the product of the two corresponding primes, triples to products of their relevant primes, etc., finally sending $\{1, \ldots, n\}$ itself to 0. Clearly for two subsets A and B of $\{1, \ldots, n\}, A \supseteq B$ iff $h(A)$ is a multiple of $h(B)$, whence h is an isomorphism between $\mathbf{H}_n$ and $\mathbf{S}_n$.

COMPLETENESS THEOREM. *Each tautology of* **MM** *is a theorem of $C5$.*

PROOF. Assume contrapositively that α is not provable in $C5$. By the first lemma, α fails in some $\mathbf{H}_n$. By the second, it therefore fails as well in the corresponding submatrix $\mathbf{S}_n$ of **MM**, whence α fails in **MM** itself.

3 Some matters for further study

Scroggs shows in [6] that the proper extensions of full $S5$ are linearly ordered by inclusion of their theorem sets, and that all have finite characteristic matrices. In contrast, the extensions of $C5$ form a far more complex lattice, one largerly awaits further investigations. Some such extensions (cf. [7]) are characterized only by infinite matrices. And not even those characterized by finite matrices are linearly ordered; for example, the logic characterized by the submatrices of **MM** with values $\{1, 2, 3, 0\}$ and $\{1, 2, 4, 0\}$ respectively, are independent of one another, each containing theorems the other lacks.

Nor do the systems characterized by the submatrices of **MM** even exhaust the extensions of $C5$. Indeed, let us call an extension of $C5$ *Hallden-complete* just in the case it contains no theorems of the form $CC\alpha\beta\beta$ where α and β have no sentence letters in common and neither α nor β is itself a theorem of the extension in question. Then, as the author plans to show in a latter paper, an extension of $C5$ is characterized by a submatrix of **MM** if and only if it is Hallden-complete.

References

[1] E. J. Lemmon, C. A. Meredith, D. Meredith, A. N. Prior and I. Thomas, *Calculi of pure strict implication mimeograph*, Philosophy Department Canterbury University College, Christchurch, New Zeland, 1957. Latter published in **Philosphical Logic**, ed. J. W. Davis, et. al., D. Reidel, Dordrecht, 1969.

[2] C. I. Lewis and C. H. Langford, **Symbolic Logic**, Dover, New York, 1932.

[3] W. T. Parry, *Zum Lewisschen Aussagenkalkül*, **Ergebnisse eines mathematischen Kolloquiums**, vol. 4 (1933).

[4] A. N. Prior, *Time and Modality*, Clarendon Press, Oxford, 1957.

[5] N. Rescher, *A probablilistic approach to modal logic*, **Acta Philosophica Fennica**, vol. 16 (1963), pp. 215–225.

[6] S. J. Scroggs, *Extensions of the Lewis system S5*, **Journal of Symbolic Logic**, vol. 16 (1951), pp. 112–120.

[7] D. Ulrich, *A descending chain of incomplete extensions of implicational $S5$*, **Zeitschrift für mathematische Logic und Grundlagen der Mathematik**, vol. 31 (1985), pp. 201–208.

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