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A TOPOLOGICAL INTERPRETATION OF DIAGONALIZABLE ALGEBRAS*

In [1] the authors show that every diagonalizable algebra (A, d) can be embedded in an algebra of the form $(2^X, d_0)$, where X is a topological space and d_0 is the derivative operation in X.

The aim of the present paper is to show that every diagonalizable algebra can be embedded into an algebra of the form $(2^X, c)$ where X is a topological space and c is the condensation operation in X.

1. A diagonalizable algebra is a system (A, d) such that A is a Boolean algebra and the unary operation d satisfies the following conditions:

i) $d0 = 0$,
ii) $d(x \vee y) = dx \vee dy$,
iii) $dx = d(x - dx)$,
iv) $ddx \leq dx$.

LEMMA 1. *For each diagonalizable algebra (D, d) there exists a diagonalizable algebra (A, s) such that*

1) (D, d) is embedable into (A, s),
2) the Boolean algebra A is a subalgebra of the algebra $2^H (A \leq 2^H)$, for some set H with $card(H) \geq \mathsf{c}$,
3) all countable subsets of the set H belong to A,
4) for all countable $X \subseteq H, s(X) = \emptyset$.

PROOF. Assume that the Boolean algebra D equals $(C, \cap, \cup, -, \emptyset)$, i.e., it is a field of subsets of a set $T, C \leq 2^T$.

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Put $H = T \times R$, where R is the set of real numbers and let $h : C \to 2^H$ be defined as follows: $h(Y) = Y \times R$, for each $Y \in C$. Observe that h is an embedding, putting $B = h(C)$ we obtain: for every $U \in B$, if $U \neq \emptyset$, then $card(U) \geq \mathsf{c}$. Moreover, one can define an operation d in B, by the isomorphism h, such that (B, d) is a diagonalizable algebra and $B \leq 2^H$.

Define a relation $\sim$ on 2^H as follows:
$U \sim Y$ iff the sets $U - Y$ and $Y - U$ are countable.
It is easy to see that $\sim$ is a congruence relation on the algebra 2^H.

Define now the family of sets A by the formula: $Y \in A$ iff $Y \sim U$ for some $U \in B$. Observe that

1) for every $Y \in A$ there exists exactly one $U \in B$ such that $Y \sim U$,

2) $(A, \cap, \cup, -, \emptyset)$ is a Boolean algebra such that all countable subsets of H belong to A,

3) $Y \in A$ iff $Y = (U - Z_1) \cup Z_2, U \in B$ and Z_i $(i = 1, 2)$ are countable.

Define the operation s on A in the following manner: for $T \in A$, $s(Y) = d(U)$, where $Y \sim U \in B$. It is easy to check that the algebra $(A, \cap, \cup, -, \emptyset, s)$ is a diagonalizable algebra and $(B, d) \leq (A, s)$. Moreover, (A, s) is atomic and for each countable $X \subseteq H, X \sim \emptyset \in B$, so $s(X) = d\emptyset = \emptyset$.

2. We now recall some topological notations. Let X be a topological space. X is T_1 space if for every pair x, y of different points of X there exists an open set V such that $x \in V$ and $y \notin V$. Equivalently, X is T_1 iff for every $x \in X$, the set $\{x\}$ is closed.

Let $A \subseteq X$. We shall say that the point $p \in X$ is an accumulation point of the set A if p belongs to the closure of $A - \{p\}$. The derivative operation d is defined as follows: $d(A)$ is the set of all accumulation points of the set A. In other words
$p \in dA$ iff for every neighborhood U_p of the point $p, U_p \cap A - \{p\} \neq \emptyset$.

If X is T_1 space, then
$p \in dA$ iff for every neighborhood U_p, the set $U_p \cap A$ is infinite.

We shall say that a point p is a condensation point of the set A if for each neighborhood U_p the set $U_p \cap A$ is uncountable (cf. [3]). Let c be the condensation operation, i.e. cA is equal to the set of all condensation points of the set A.

The operation c fulfills the following conditions:

i) $c\emptyset = \emptyset$,
ii) $c(Y \cup Z) = cY \cup cZ$,
iii) $ccY \subseteq cY$.

Moreover, $cY \subseteq dY \subseteq Cl(Y) = Y \cup dY$, where Cl denotes the closure operation.

Let (A, s) be the diagonalizable algebra, which satisfies the conditions of the lemma 1. Recall that $A \leq 2^H$. We define a topology over the set H as follows (cf. [1]). For every $Y \subset H$,

$$Cl(Y) = \bigcap \{Z \cup sZ : Y \subseteq Z \cup sZ, Z \in A\}.$$

Equivalently, the family of sets $\{Z - s(-Z) : Z \in A\}$ is a base for the topology.

CLAIM 1. *For every $X \in A$ we have $sX = dX$, where d is the derivative operation.*

For a proof see [1].

CLAIM 2. *The topological space H is T_1.*

PROOF. Let $h \in H$, then $Cl\{h\} = \{h\} \cup d\{h\}$. On the other hand $d\{h\} = s\{h\} = \emptyset$. Hence $Cl\{h\} = \{h\}$.

We shall show that in the topological space H the operation d and c are equal. Therefore one can say that in the diagonalizable algebra (A, s) the operation s may be interpreted as the condensation one in some topological space.

LEMMA 2. *Assume that in a topological space all countable sets are closed. Then the derivative operation is equal to the condensation one.*

PROOF. As above, let dA be the derivative set of A and cA be the condensation set of A. We shall show that

$$\text{if } p \in dA, \text{ then } U_p \cap A \text{ is uncountable for every } U_p.$$

Assume that there is an U_p such that $U_p \cap A$ is countable, thus the set $U_p \cap A - \{p\}$ is countable too and it is closed. Let $V_p = U_p - (U_p \cap A - \{p\})$, then $p \in V_p$. But $p \in dA$, so $V_p \cap A - \{p\} \neq \emptyset$. On the other hand $V_p \cap A - \{p\} = \emptyset$, a contradiction. In any topological space $cA \subseteq dA$, which ends the proof.

Observe that in the topological space H for each countable $X \subseteq H$ we have $dX = sX = \emptyset$. Hence $Cl(X) = X \cup dX = X$. Therefore every countable set is closed.

THEOREM. *For any diagonalizable algebra (A, d) there exists a T_1 topological space H such that the algebra (A, d) is embedable into $(2^H, c)$, where c is the condensation operation in H.*

3. The theorem can be generalized as follows. Let X be a topological space and let $A \subseteq X$. A point $p \in X$ is a condensation point of the rank m of the set A (shortly: m-condensation point) if for every neighborhood V_p, the cardinality of the set $V_p \cap A$ is not less then m, i.e. $card(V_p \cap A) \geq m$, cf. [3].

The generalized version of the theorem is obtained if we replace the notion "c is the condensation operation" by "c is the m-condensation operation".

4. Let c be the condensation operation in a topological space X. Assume that X is scattered with respect to c (shortly c-scattered), i.e. the following condition holds

$$Z \subseteq cZ \Longrightarrow Z = \emptyset, \text{ for every } Z \subseteq X.$$

Then the algebra $(2^X, c)$ is a diagonalizable algebra and

(*) for any countable $Z \subseteq X, c(X) = \emptyset$.

Conversely, let $(2^X, d)$ be a diagonalizable algebra which satisfies the condition (*) for the operation d. Then there exists a topology over the set X such that d equals c – the condensation operation in X and X is c-scattered. For a proof (cf. [2]) consider an operator $C : 2^X \to 2^X$ defined as follows:

$$C(Z) = Z \cup dZ, \text{ for any } Z \subseteq X.$$

It is easy to show that C is a closure operator. Hence the topology over X is defined. Let d_0 denotes the derivative operation in this topology. As in

[2] one can show that

$$d_0 Z = dZ \text{ for every } Z \subseteq X.$$

Each countable set $A \subseteq X$ is closed since for such sets $dA = \emptyset$. Hence by Lemma 2 the derivative operation and the condensation one are equal on X. The space X is c-scattered, for the algebra $(2^X, d)$ is diagonalizable.

References

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