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Ewa Graczyńska

ON SOME OPERATORS ON PSEUDOVARIETIES

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1. Pseudovarieties

Birkhoff's theorem [1] asserts that a class of algebras of a given type $\tau : T \to N$ is a *variety* if and only if it is closed under operation of passing to subalgebras, homomorphic images and also arbitrary direct products. Motivated by applications of the theory of automata Eilenberg and Schützenberger considered in [2], [3] classes of finite monoids closed under subalgebras, homomorphic images and finite products. These considerations led to the notion of a *pseudovariety* defined in [3] as follows:

Definition 1. (Eilenberg, Schützenberger)
A class V of finite algebras is called a pseudovariety if the following conditions hold:

(i) If $\mathbf{A} \in V$ and $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ then $\mathbf{B} \in V$.
(ii) If $\mathbf{A} \in V$ and $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$, then $\mathbf{B} \in V$.
(iii) If $\mathbf{A}, \mathbf{B} \in V$, then $\mathbf{A} \times \mathbf{B} \in V$.

From now on, following motivations from automata theory ([2], [8]) we consider only finite algebras of finite type, i.e. such that T is a finite set. Therefore, any algebra $\mathbf{A}$ of a given type τ can be written as: $\mathbf{A} = (A, f_1, \ldots, f_n)$ where A is a finite set and $\{f_1, \ldots, f_n\}$ denote the family of fundamental operations on $\mathbf{A}$.

DEFINITION 2. (Eilenberg, Schützenberger)
Given a sequence $\Sigma = (e_n : n \in N)$ *of equation of type* τ, *we say a class* V *of algebras of type* τ *is ultimately defined by* Σ *if and only if any algebra* $\mathbf{A}$ *of* V *satisfies all but a finite number of equations of* Σ.

The following theorem was proved in [3], [8] for semigroups (monoids). However it was pointed in [3] that it holds for varieties of algebras of any finite type (i.e. such that T is a finite set). To make the paper self-contained, we write a proof of this theorem, which is a slight modification of that of [3], modified as required by the authors of [3]:

THEOREM 1. (Eilenberg, Schützenberger)
Each nonempty pseudovariety is ultimately defined by a sequence of equalities.

PROPOSITION 1. (Eilenberg, Schützenberger)
A finite congruence $\sim$ *in* Ξ_n *is finitely generated, i.e. there exists a finite set* $W \subseteq \Xi_n \times \Xi_n$ *such that* $u \sim v$ *for all* $(u, v) \in W$ *and such that* $\sim$ *is the smallest congruence with this property.*

2. The Lattice of Pseudovarieties

From now on we consider pseudovarieties of a given type τ. If V is a pseudovariety ultimately defined be a sequence Σ of equalities of type τ, then we write $\Sigma \to V$. If $\Sigma_1 = (a_n : n \in N)$, $\Sigma_2 = (b_n : n \in N)$ then $\Sigma_1 \sqcup \Sigma_2$ denotes the sequence $(a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots : n \in N)$.

According to [3], [8] the family $L(\tau)$ of all pseudovarieties of a given type form a complete lattice $\mathbf{L} = (L(\tau), \vee, \cap)$. This lattice $\mathbf{L}$ can be also written as a p.o. set $\mathbf{L} = (L(\tau), \subseteq)$. In the sequel $\Sigma_{\mathbf{f}}$ or $\Sigma_{\mathbf{f}_i}$ denotes always a finite subsequence of a given sequence Σ or Σ_i, for $i = 1, 2$.

If $\Sigma = (a_n : n \in N)$ is a sequence of equalities of a given type, then $E(\Sigma)$ denotes the closure of the set $\Sigma = \{a_n : n \in N\}$ under the Birkhoff's rules of inferences (see [1], [4]). For an algebra $\mathbf{A}$, and a set Σ of equalities of a given type, we write $\mathbf{A} \vdash \Sigma$ provided $\mathbf{A}$ satisfies all equalities of Σ.

PROPOSITION 2. *Let* V_i *be a pseudovariety of type* τ, *for* $i = 1, 2$ *and* $\Sigma_i \to V_i$, *for* $i = 1, 2$. *Then* *(iv)* $\Sigma_1 \sqcup \Sigma_2 \to V_1 \cap V_2$,

(v) $\mathbf{A} \in V_1 \vee V_2$ *iff* $\mathbf{A} \vdash E(\Sigma_1 - \Sigma_{\mathbf{f}_1}) \cap E(\Sigma_2 - \Sigma_{\mathbf{f}_2})$, *for some* $\Sigma_{\mathbf{f}_i} \subseteq \Sigma_i$, $i = 1, 2$.

Remark 1. *An analogous property of finite joins of pseudovarieties can be formulated.*

3. Normal and regular pseudovarieties

Our aim is to extend some ideas of [5] – [10] for pseudovarieties (without constants). Firstly we recall the notion of *normal* and *regular* equalities investigated by I. I. Mel'nik [7], J. Płonka [9] and considered by several authors (see [5] – [10]). An *equality* $p = q$ of a given type τ is called *normal* if neither p nor q is a variable or p and q are the same variable. Otherwise $p = q$ is *not-normal.* An equality $p = q$ is *regular* iff $Var(p) = Var(q)$, where $Var(p)$ denotes the set of all variables occurring in term p. Otherwise $p = q$ is *nonregular.* Denote by $N(\tau), R(\tau)$ the set of all normal (regular) equalities of type τ. A *variety* is normal (regular) if it is defined by a set of normal (regular) equalities. Otherwise, it is *non-normal* (*nonregular*).

Motivated by several applications of the notions of normal and regular equational theories, especially in some constructions of algebras (see [7], [9]) or lattice of equational theories (see [5]) we extend the notion of normal and equational theories for pseudovarieties. Several examples of such structures were considered by J. E. Pin [8].

Definition 3. *A pseudovariety V is called normal (regular) if there exists a sequence Σ of normal (regular) equalities ultimately defining V. Otherwise V is not-normal (nonregular).*

Following [6], [7], [9] and [10] we express the property that a pseudovariety V is normal (regular) by suitable operations on algebras from V. Given an algebra $\mathbf{A} = (A, f_1, f_2, \dots, f_n)$. Let O be a new element, i.e. $O \notin A$. Let $A_* = A \cup \{O\}$. Then $\mathbf{A}_* = (A_*, f_{1*}, \dots, f_{n*})$, where: $f_{k*}(a_1, \dots, a_{n_k}) = O$ for all elements $a_1, \dots, a_n \in A_*$. $\mathbf{A}_*$ is called a *zero-algebra* of an algebra $\mathbf{A}$. It is easy to see that $E(\mathbf{A}_*) = N(\tau)$, where $E(\mathbf{B})$ denotes the set of all equations satisfied in an algebra $\mathbf{B}$. In [10] the notion of a *sup-algebra* $\mathbf{A}^*$ of an algebra $\mathbf{A}$ was defined. Similarly as in [10] we deduce:

PROPOSITION 3. *A pseudovariety V is normal (regular) if and only if it is closed under formation of zero-algebras (sup-algebras).*

REMARK 2. *By Propositions 2, 3 and Remark 1 it follows that the family of all normal (regular) pseudovarieties form a complete sublattice of the lattice* $\mathbf{L}$.

PROPOSITION 4. *A pseudovariety V is not-normal (nonregular) iff for every sequence Σ of equalities, ultimately defining V, there exists an infinite subsequence Ψ of Σ, such that Ψ contains only not-normal (nonregular) equations.*

4. Operators N and R on pseudovarieties

We introduce two operators on pseudovarieties. As we show that the lattice of equational theories (varieties of a given type τ) form a sublattice (up to isomorphism) of the lattice $\mathbf{L}$ of pseudovarieties of a given type, therefore we obtain a natural generalization of operators considered in [5]. We examine some properties of these operators and show a generalization of some results of [5].

THEOREM 2. *Let $\mathcal{L}$ denotes the lattice $(\mathcal{L}(\tau), \cap, \vee)$ of all varieties of type τ. Then $\mathcal{L}$ is a sublattice of $\mathbf{L}$ up to isomorphism.*

Given a pseudovariety V, then $\mathbf{N}(V)$ and $\mathbf{R}(V)$ denote the family of all normal (regular) pseudovarieties (of a given type) containing V, respectively. We define operators N and R as follows:

DEFINITION 4. *For a pseudovariety $V \in L(\tau)$, $N(V) = \bigcap \mathbf{N}(V), R(V) = \bigcap \mathbf{R}(V)$.*

REMARK 3. *From Remark 2 it follows, that for a pseudovariety $V \in L(\tau)$, $N(V)$ $(R(V))$ is the smallest normal (regular) pseudovariety, such that $V \subseteq N(V)$ $(V \subseteq R(V))$, respectively. Moreover, V is normal (regular) iff $V = N(V)$ $(V = R(V))$.*

PROPOSITION 5. *Let $V \in L(\tau)$ and $\Sigma \to \mathbf{V}$. Then:*

(vi) $\mathbf{A} \in N(V)$ *iff* $\mathbf{A} \vdash E(\Sigma - \Sigma_{\mathbf{f}}) \cap N(\tau)$, *for some* $\Sigma_{\mathbf{f}} \subseteq \Sigma$,

(vii) $\mathbf{A} \in R(V)$ *iff* $\mathbf{A} \vdash E(\Sigma - \Sigma_{\mathbf{f}}) \cap R(\tau)$, *for some* $\Sigma_{\mathbf{f}} \subseteq \Sigma$.

THEOREM 3. *For every $V \in L(\tau), V = N(V)$ $(V = R(V))$ or the pseudovariety $N(V)$ $(R(V))$ covers V in the lattice* $\mathbf{L}$.

THEOREM 4. *The operator $N : \mathbf{L} \to \mathbf{L}$ is an endomorphism.*

THEOREM 5. *If V is a not-normal pseudovariety, then the operator N is an embedding of the lattice $\mathbf{L}(V)$ into the lattice $\mathbf{L}(N(V))$.*

PROPOSITION 6. *Let V, K be pseudovarieties and $K \subseteq N(V)$. Then there are only two possibilities: $K \subseteq V$ or K is normal.*

THEOREM 6. *Given a not-normal pseudovariety $V \in L(\tau)$. Then the lattice $\mathbf{L}(N(V))$ is isomorphic to the direct product of the lattice $\mathbf{L}(V)$ and 2.*

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Institute of Mathematics
Polish Academy of Sciences
ul. Kopernika 18, 51-617 Wrocław
Poland