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COMMENTS ON A QUESTION OF WOLNIEWICZ

In the preceding issue of this **Bulletin** (vol. 19, no 3, p. 108) Professor Wolniewicz posed the following question about join-semilattices L with unit:

"Under what properties of L – short of all its maximal ideal being finite – does an antichain $MinV(A)$ always contain a minimal subchain B such that $r(B) = r(MinV(A))$?"

The very formulation of the question indicates that Wolniewicz knows the following answer which we write down as

Theorem 1. *Let L be a non-degenerate join-semilattice with unit. Assume that each maximal ideal of L is finite. Then every antichain of the form* $M = MinV(A)$ *contains a minimal subset* $B \subseteq M$ *such that* $r(B) = r(M)$.

1. In this section we make comment on Wolniewicz's answer. We first note that Theorem 1 can be stated in a bit more general form:

Theorem 1'. *Let L be a join-semilattice with unit. Assume that each maximal ideal of L is finite. Then for every* $M \subseteq L$ *there is* $B \subseteq M$ *such that* $r(B) = r(M)$ *and B is minimal with respect to this property.*

Proof. Define $\mathcal{B} = \{B \subseteq M : r(B) = r(M)\}$. (Recall that $r(M) = \{R : M \cap R \neq \emptyset$ and R is a maximal ideal of $L\}$.) We check that $\mathcal{B}$ is closed under arbitrary intersections of chains and the partial order $(\mathcal{B}, \supset)$ fulfills the assumption of Kuratowski-Zorn's Lemma. Then a maximal element in $(\mathcal{B}, \supset)$ is a minimal one in $\mathcal{B}$. □

To grasp the idea of Theorem 1 we formulate now several equivalents of the assumption to the effect that every maximal ideal is finite.

PROPOSITION 1. *Let L be a join-semilattice with unit. Then the following four conditions are equivalent:*

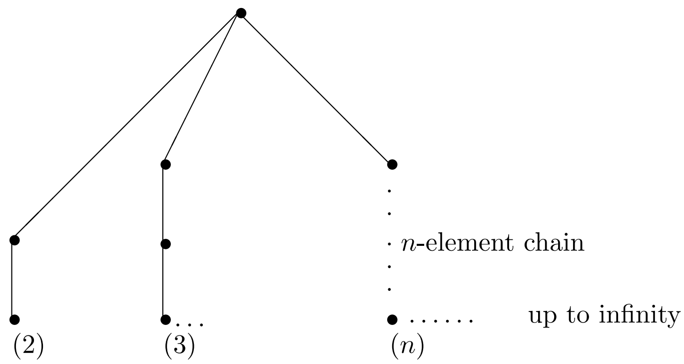
(i) Every proper ideal of L is finite;

(ii) Every maximal ideal of L is finite;

(iii) Every proper ideal of L is principal and finite;

(iv) L satisfies ACC (the ascending chain condition) and every proper and principal ideal of L is finite. □

Notice that L satisfies ACC iff every (nonempty) ideal of L is principal. Moreover, none of the conditions (i)-(iv) implies the finiteness of L as shown by the semilattice represented by the following diagram:



2. At this point we wish to discuss the notion of minimality in $\mathcal{B}$-like families of sets. We get rid of algebraic assumptions, and therefore, we let L be an arbitrary non-empty set and $\mathcal{R}$ be a non-empty family of non-empty subsets of L. Then we define for $M \subset L$

$$r(M) = \{R \in \mathcal{R} : R \cap M \neq \emptyset\}$$

and

$$\mathcal{B} = \{B \subset M : r(B) = r(M)\}.$$

PROPOSITION 2. *Under the above assumptions the following conditions are equivalent:*

(a) There is a minimal element in $(\mathcal{B}, \subset)$*;*
(b) There is a $B_0 \in \mathcal{B}$ *such that:*

(*) $$(\forall_{b \in B_0})(\exists_{R \in \mathcal{R}})(R \cap B_0 = \{b\}).$$

Proof. $(a) \to (b)$. Let B_0 be a minimal element in $(\mathcal{B}, \subset)$. Then $r(B_0) = r(M)$ and $r(B') \neq r(B_0)$ for all $B' \subset B_0$ such that $B' \neq B_0$. In particular, for every $b \in B_0$, $r(B_0 - \{b\}) \neq r(B_0)$. Hence there is $R \in \mathcal{R}$ such that $R \cap B_0 \neq \emptyset$ and $R \cap (B_0 - \{b\}) = \emptyset$. So $R \cap B_0 = \{b\}$ which proves that our B_0 satisfies condition (*).

$(b) \to (a)$. Let $B_0 \in \mathcal{B}$ fulfill (*). Suppose that B_0 is not minimal in $(\mathcal{B}, \supset)$. Hence there is $B' \subset B_0$ such that $r(B') = r(M)$ and $B' \neq B_0$. Take $b \in B_0$ such that $b \notin B'$. Since $B' \subset B_0 - \{b\} \subset B_0$ we have $r(B') \subset r(B_0 - \{b\}) \subset r(B_0)$ which contradicts the hypothesis that B_0 fulfills (*). As a result B_0 is minimal. □

Proposition 2 furnishes an answer to Wolniewicz's question when we take $\mathcal{R}$ to be a set of all maximal ideals in a non-degenerate join-semilattice L with unit. Another answer is given by the following corollary:

Corollary. *Let* $M \subset L$*, and let* $1 \notin M$ *where* 1 *is the unit of* L*. Assume that a set* $B_0 \in \mathcal{B}$ *satisfies condition:*

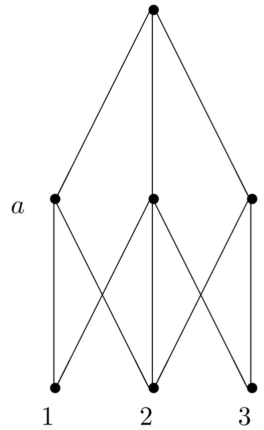
(**) $$(\forall_{b, b' \in B_0})(b \neq b' \to b \vee b' = 1).$$

Then B_0 *is minimal in* $\mathcal{B}$*.*

Proof. We shall prove that our set B_0 satisfies the separation condition (*) of Proposition 2. Let then $b \in B_0$, and let $(b]$ be the principal ideal generated by b. Since $1 \notin M$, $(b]$ is proper. Hence there is a maximal ideal R which extends $(b]$. We show that $R \cap B_0 = \{b\}$. Suppose then that there is $b' \neq b$ such that $b' \in R \cap B_0$. So $b \vee b' \in R$. But by (**), $b \vee b' = 1$, which means that R is improper. This contradiction proves that our B_0 satisfies condition (*) of Proposition 2, and thus B_0 is minimal.

3. At this point we furnish two examples. Firstly, we give a counterexample to Wolniewicz's parenthetic remark: "Actually we have in general: if $B \subset A$ and $r(A) = r(B)$ then $r(A - B) = r(B)$."

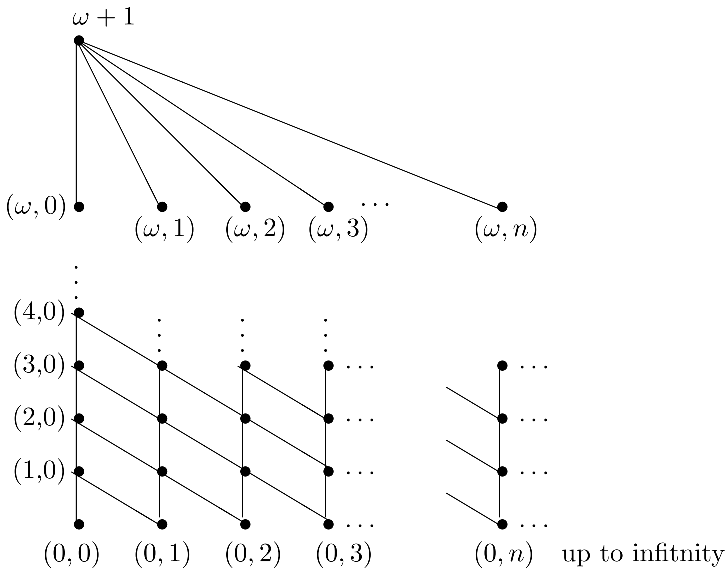
Let L be a join-semilattice represented by the diagram below, and let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$.



We have $r(A) = r(B)$, but $r(A - B) \neq r(B)$ since $(a] \notin r(A - B)$.

Secondly, we give an example of a join-semilattice F such that the family $\mathcal{B} = \{B \subset M : r(B) = r(M)\}$, where M is the set of *all* minimal elements of F, has *no* minimal subset. The example is due to T. Furmanowski who has kindly agreed to state it here.

Furmanowski's semilattice F is represented by the following diagram.



It should be easily seen that the universe of F is identical with the set $((\omega \cup \{\omega\}) \times \omega) \cup \{\omega + 1\}$ and

(1) R is a maximal ideal of F iff R is a principal ideal generated by an element of the form (ω, n) for $n = 0, 1, 2, \ldots$

(2) $M = \{(0, n) : n \in \omega\}$ is the set of all minimal elements of F and, moreover, $r(M) =$ the set of all maximal ideals of F;

(3) For every $B \subset M, r(B) = r(M)$ iff B is infinite.

The above conditions imply that the family $\mathcal{B}$ has no minimal element.

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