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TOPOS BASED SEMANTICS FOR CONSTRUCTIVE LOGIC WITH STRONG NEGATION

The theory of elementary toposes plays the fundamental role in the categorial analysis of the intuitionistic logic. The main theorem of this theory uses the fact that sets $E(A, \Omega)$ (for any object A of a topos E) are Heyting algebras with operations defined in categorial terms. More exactly, subobject classifier true: $1 \to \Omega$ permits us define truth-morphism on Ω and operations in $E(A, \Omega)$ are defined by them uniformly.

The aim of this paper is to show usefulness of toposes in the categorial analysis of the constructive logic with strong negation ($CLSN$, for short) too. In any topos E we distinguish an object Λ and its truth-arrows that the sets $E(A, \Lambda)$ have the structure of a Nelson algebra. The object Λ (internal Nelson algebra) in E is defined as a result of an application, to the internal Heyting algebra Ω, the topos counterpart of the well-known classical construction of the Nelson algebra $N(B)$ for a given Heyting algebra B, (see [1], [4] for a generalization).

We denote by HA the variety of all Heyting algebras and by NA the variety of all Nelson algebras. Explanations of definitions and notations of used notions from topos theory are in [2]. Truth-morphisms are denoted like respective connectives.

1. Object Λ and its truth-arrows

Let E be an elementary topos. We shall write Ω^4 instead of $\Omega \times \Omega \times \Omega \times \Omega$ and projections from Ω^4 to Ω will be denoted by pr_i^4 for $i = 1, 2, 3, 4$. Of course $\Omega^4 \cong (\Omega \times \Omega)^2$ and projections defining this product we denote analogously by pr_i^2 $(i = 1, 2)$. It is obvious that $< pr_1^4, pr_2^4 > = pr_1^2$ and $< pr_3^4, pr_4^4 > = pr_2^2$.

Let $(\Lambda, \lambda : \Lambda \to \Omega \times \Omega)$ be an equalizer of a pair $(\cap, false_{\Omega\times\Omega})$. Of course $false_{\Omega\times\Omega} \cdot f = false_A$ for any $f : A \to \Omega\times\Omega$. We shall use morphism $\lambda^2 : \Lambda \times \Lambda \to \Omega^4$ which is defined as follows: $\lambda^2 = < \lambda \cdot pr_1, \lambda \cdot pr_2 >$ $(pr_1, pr_2 : \Lambda \times \Lambda \to \Lambda$ define product $\Lambda \times \Lambda)$.

LEMMA 1. *Each of morphisms $\beta_\wedge, \beta_\vee, \beta_\to : \Omega^4 \to \Omega \times \Omega$ defined by compositions:*

$\beta_\wedge := < pr_1^4 \cap pr_3^4, pr_2^4 \cup pr_4^4 > \cdot\lambda^2$
$\beta_\vee := < pr_1^4 \cup pr_3^4, pr_2^4 \cap pr_4^4 > \cdot\lambda^2$
$\beta_\to := < pr_1^4 \Rightarrow pr_3^4, pr_1^4 \cap pr_4^4 > \cdot\lambda^2$

as well as each of $\beta_\neg, \beta_\sim : \Lambda \to \Omega \times \Omega$ defined by

$\beta_\neg := < -pr_1, pr_1 > \cdot\lambda$
$\beta_\sim := < pr_2, pr_1 > \cdot\lambda$

equalize the pair $(\cap, false_{\Omega\times\Omega})$

PROOF. Equation $\cap\cdot\beta_\wedge = false_{\Lambda\times\Lambda}$ is a conclusion of the after-mentioned computation. All equations hold because $E(\Lambda\times\Lambda, \Omega)$ is a Heyting algebra.

$$\begin{aligned}\cap \cdot \beta_\wedge = {} & ((pr_1^4 \cap pr_3^4) \cdot \lambda^2) \cap ((pr_2^4 \cup pr_4^4) \cdot \lambda^2)\\ = {} & (pr_1^4 \cdot \lambda^2 \cap pr_3^4 \cdot \lambda^2) \cap (pr_2^4 \cdot \lambda^2 \cup pr_4^4 \cdot \lambda^2)\\ = {} & (pr_1^4 \cdot \lambda^2 \cap pr_3^4 \cdot \lambda^2 \cap pr_2^4 \cdot \lambda^2) \cup (pr_1^4 \cdot \lambda^2 \cap pr_3^4 \cdot \lambda^2 \cap pr_4^4 \cdot \lambda^2)\\ = {} & (pr_1^4 \cdot \lambda^2 \cap pr_2^4 \cdot \lambda^2 \cap pr_3^4 \cdot \lambda^2) \cup (pr_1^4 \cdot \lambda^2 \cap pr_3^4 \cdot \lambda^2 \cap pr_4^4 \cdot \lambda^2)\\ = {} & ((\cap \cdot \lambda \cdot pr_1) \cap pr_3^4 \cdot \lambda^2) \cup (pr_1^4 \cdot \lambda^2 \cap (\cap \cdot \lambda \cdot pr_2))\\ = {} & (false_{\Lambda\times\Lambda}(pr_3^4 \cdot \lambda^2) \cup (pr_1^4 \cdot \lambda^2 \cap false_{\Lambda\times\Lambda})\end{aligned}$$

(The last equation holds because $\cap \cdot \lambda = false_\Lambda$)

$$= false_{\Lambda\times\Lambda} \cup false_{\Lambda\times\Lambda} = false_{\Lambda\times\Lambda}.$$

The proof of $\cap \cdot \beta_\vee = false_{\Lambda\times\Lambda}$ is analogous to the one presented for conjunction.

Of course $\cap \cdot \beta_\to = (pr_1^4 \cdot \lambda^2 \Rightarrow pr_3^4 \cdot \lambda^2) \cap (pr_1^4 \cdot \lambda^2 \cap pr_4^4 \cdot \lambda^2)$. But in the Heyting algebra $E(\Lambda \times \Lambda, \Omega)$: $(pr_1^4 \cdot \lambda^2 \Rightarrow pr_3^4 \cdot \lambda^2) \cap pr_1^4 \cdot \lambda^2 \leq pr_3^4 \cdot \lambda^2$.

Thus $(pr_1^4 \cdot \lambda^2 \Rightarrow pr_3^4 \cdot \lambda^2) \cap (pr_1^4 \cdot \lambda^2 \cap pr_4^4 \cdot \lambda^2) \leq pr_3^4 \cdot \lambda^2 \cap pr_4^4 \cdot \lambda^2$ and $pr_3^4 \cdot \lambda^2 \cap pr_4^4 \cdot \lambda^2 = \cap \cdot \lambda \cdot pr_2 = false_{\Lambda\times\Lambda}$ and $\cap \cdot \beta_\to = false_{\Lambda\times\Lambda}$. It is clear that $\cap \cdot \beta_\neg = \cap \cdot \beta_\sim = false_{\Lambda\times\Lambda}$. $\square$

Now we can define truth-morphism $\wedge, \vee, \rightarrow: \Lambda \times \Lambda \rightarrow \Lambda$ and $\neg, \sim: \Lambda \rightarrow \Lambda$ as liftings of the respective β's along λ, i.e. $*$ for $* \in \{\wedge, \vee, \rightarrow, \neg, \sim\}$ is the unique morphism such that $\lambda \cdot * = \beta_*$. Because $false \ \cap \ true = true \ \cap \ false = false$ we can distinguish two morphisms $T, F : \Lambda \rightarrow \Lambda$ which are liftings of $<true, false>$ and $<false, true>$ respectively along λ.

2. Nelson algebra structure of $E(A, \Lambda)$

Let A be an arbitrary object of a topos E. For any $f, g \in E(A, \Lambda)$ we define:

$f \wedge g := \wedge \cdot < f, g >$
$f \vee g := \vee \cdot < f, g >$
$f \rightarrow g := \rightarrow \cdot < f, g >$
$\neg f := \neg \cdot f$
$\sim f := \sim \cdot f$
$F_A := F \cdot !_A$
$T_A := T \cdot !_A$

THEOREM 1. *The algebra* $<E(A, \Lambda), \wedge, \vee, \rightarrow, \neg, \sim, F_A, T_A>$ *is a Nelson algebra isomorphic to the algebra* $N(<E(A, \Omega), \cap, \cup, \Rightarrow, -, false_A, true_A>)$.

PROOF. If is sufficient to verify that the map $H : E(A, \Lambda) \rightarrow N(E(A, \Omega))$ defined as follows:

$$H(h) = (pr_1 \cdot \lambda, pr_2 \cdot \lambda \cdot h)$$

is a required isomorphism. $\Box$

EXAMPLES. (1) In topos Set we have $\Omega = \{0, 1\}$ and all operations $\cap, \cup, \Rightarrow$, $-$ are usual operations in Boolean algebra $2 = < \{0, 1\}, \cap, \cup, \Rightarrow, -, 0, 1 >$. It is easy to see that $\Lambda = \{(0, 1), (0, 0), (1, 0)\}$ and truth-morphism establish a structure of Nelson algebra on Λ. Of course in Set, $Set(1, A) \cong A$ and $N(Set(1, \Omega)) \cong N(2) \cong Set(1, \Lambda)$.

(2) We shall consider topos Set^P for any poset P. In Set^P, $\Omega : P \rightarrow Set$ is defined by components $\Omega_P = [p)^+$ (Ω_P is equal to set of all hereditary sets of $[p) = \{q \in P : p \leq q\}$). Truth-morphisms are defined by components too. For example $\cap_P : \Omega_P \times \Omega_P \longrightarrow \Omega_P$ is a set-theoretical intersection. Now it is easy to see that $\Lambda : P \rightarrow Set$ is defined by components

$$\Lambda_P = \{< S, T > \in \Omega_P \times \Omega_P : S \cap T = 0\}.$$

3. Completeness theorem

Let Φ_0 be a set of variables $\{x_0, x_1, \ldots\}$ and Φ be a set of all formulas built up using of the connectives of $CLSN$. We shall say that any sentence $\alpha \in \Phi$ is true on a valuation $V : \Phi_0 \to E(1, \Lambda)$ if $\widetilde{V} = T$, where $\widetilde{V} : \Phi \to E(1, \Lambda)$ is an usual extension of V. Any sentence $\alpha \in V$ is valid in topos E iff $\widetilde{V} = T$ for any valuation V. This fact we denote by $E \models_\Lambda \alpha$. One have noticed that we simply define an usual semantics in a Nelson algebra $E(1, \Lambda)$.

THEOREM 2. *Let $\alpha \in \Phi$. The following conditions are equivalent:*
(i) α is tautology of $CLSN$
(ii) $E \models_\Lambda \alpha$, for any topos E
(iii) $Set^P \models_\Lambda \alpha$, for any poset P
(iv) $Set^{P_{IL}} \models_\Lambda \alpha$, where P_{IL} is the canonical frame for the intuitionistic logic.

PROOF. In the proof we shall use the following facts:

(1) [1], [3], [4]. For any $\alpha \in \Phi$ the following conditions are equivalent:
(i) α is a tautology of $CLSN$
(ii) α is valid in every Nelson algebra
(iii) α is valid in every Nelson algebra of the form $N(B)$ for every Heyting algebra B.

In particular, the least subvariety of NA containing all algebras of the form $N(B)$ ($B \in HA$) is equal to NA.

(2) [2] Let β be a sentence in the language of intuitionistic logic. Then the following conditions are equivalent:
(i) β is a tautology of intuitionistic logic
(ii) $P_{IL} \models \beta$
(iii) $Set^{P_{IL}} \models \beta$.

(2) Let capital letters H, S, P, I denote usual operators of homomorphic images, subalgebras, products and isomorphisms respectively. For any class K of Heyting algebras we have $HSP(N(K)) = IS(N(HSP(K)))$. Hence if K is a variety then $HSP(N(K)) = IS(N(K))$.

It is obvious that all implications except (iv) $\Rightarrow$ (i) hold.

Proof (iv) $\Rightarrow$ (i). Let $A = \{P_{IL}\}$. $HSP(N(A)) = IS(N(HSP(A))) = IS(N(HA)) = HSP(N(HA)) = NA$. $\square$

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