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THE QUARREL THEOREM
First attempt to the logic of lie

0. Introduction

Most of us knows these funny stories about a strange island inhabited by Truthtellers who always tell the truth and Liar who always lies. Suppose that somehow we get into this island. We meet an inhabitant and we would like to know who is he. What question answerable by YES or NO should we ask? Well, it is easily seen that one of the questions which work is following: "What could you answer if I asked you whether you are a Truthteller?"

This story expresses one of the properties of the Logic of Lie. (i.e. $\vdash ((a \lhd (a \lhd \phi)) \leftrightarrow \phi)$. The idea of construction of the Logic of Lie was inspired by two Raymond Smullyan's books ([1], [2]) which contain incredible collection of brilliant puzzles and paradoxes related to deep and important concepts of logic. One of such concepts is the well known "Liar's Paradox" which has been already put on very rich literature. This paper is also related to the "Liar's Paradox" which can be expressed by a formula elementary for the Logic of Lie.

In this paper we only describe the language of the Logic of Lie as a non-classical Sentential Calculus. Besides five standard logical connectives, namely $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, we introduce three new symbols, namely: $T, L, \lhd$. Two types of variables are used: on type – for atomic formulas, next type – for objects.

As we mentioned earlier, the "Liar's Paradox" is represented as a formula in some sense elementary for the Logic of Lie, namely the formula $a \lhd L(a)$ ("a states that a is a Liar"). This formula is obviously inconsistent. Below we consider an arbitrary set of such elementary formulas, i.e.

formulas of the form $a \lhd T(b)$ or $a \lhd L(b)$ (we can think of such a set as of a quarrel concerning veracity of the quarreling objects). We formulate and prove the theorem which provides combinatorial condition of consistency of such a set.

1. The language of the Logic of Lie

Let us describe a formal language $\mathcal{L}$. The language $\mathcal{L}$ contains:
- eight symbols called logical connectives: $T, L, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \lhd$;
- a denumerable set of symbols called atomic formulas $S_0 = \{p_1, p_2, \ldots\}$;
- a denumerable set of symbols called objects $O = \{o_1, o_2, \ldots\}$.

The notion of a formula of $\mathcal{L}$ is given by the following recursive rules:
$F0$: An atomic formula is a formula.
$F1$: If o is an object symbol then $T(o)$ and $L(o)$ are formulas.
$F2$: If ϕ is a formula so is $\neg\phi$.
$F3$: If ϕ, ψ are formulas so are $\phi \wedge \psi, \phi \vee \psi, \phi \rightarrow \psi, \phi \leftrightarrow \psi$.
$F4$: If o is an object symbol and Φ is a formula then $o \lhd \phi$ is a formula.

Let S be the set of all formulas of $\mathcal{L}$. Using the Boolean algebra $\mathcal{A} = \{\{0,1\}, -, \wedge, \vee, \rightarrow, \leftrightarrow\}$ we define the notion of a valuation.

Definition. By a *valuation* we mean an arbitrary function $V : O \cup S \rightarrow \{0,1\}$ which for every formulas ϕ, ψ and every object o satisfies the following conditions:

$$
\begin{aligned}
V0: \ & V(T(o)) = V(o) \\
& V(L(o)) = -V(o) \\
V1: \ & V(\neg\phi) = -V(\phi) \\
V2: \ & V(\phi \wedge \psi) = V(\phi) \wedge V(\psi) \\
& V(\phi \vee \psi) = V(\phi) \vee V(\psi) \\
& V(\phi \rightarrow \psi) = V(\phi) \rightarrow V(\psi) \\
& V(\phi \leftrightarrow \psi) = V(\phi) \leftrightarrow V(\psi) \\
V3: \ & V(o \lhd \phi) = V(T(o)) \leftrightarrow V(\phi)
\end{aligned}
$$

It is easy to see that every valuation is completely determined by its value on the set of objects symbols and atomic formulas. Hence we have:

PROPOSITION 1. *For any two valuations V_1 and V_2 if $V_1\lceil_{O\cup S_0} = V_2\lceil_{O\cup S_0}$ then $V_1 = V_2$.*

Let $\{q_1, q_2, \ldots, q_s\}$ be a finite subset of $O \cup S_0$. If ϕ is built up only of the elements of this set (not necessarily all) then we write $\phi = \phi(q_1, q_2, \ldots, q_s)$. The set of all such formulas is denoted by $S(q_1, q_2, \ldots, q_s)$. It is obvious that the values of an arbitrary valuation on the set $\mathcal{S}(q_1, q_2, \ldots, q_s)$ depend only of its values on the set $\{q_1, q_2, \ldots, q_s\}$. Hence

PROPOSITION 2. *For any two valuations V_1 and V_2 if $V_1\lceil_{\{q_1,q_2,\ldots,q_s\}} = V_2\lceil_{\{q_1,q_2,\ldots,q_s\}}$ then $V_1\lceil_{S(q_1,q_2,\ldots,q_s)} = V_2\lceil_{S(q_1,q_2,\ldots,q_s)}$.*

It turns out that the notion of a valuation sequence is more convenient than the notion of a valuation.

DEFINITION. A *valuation sequence* of a formula $\phi = \phi(q_1, q_2, \ldots, q_s)$ (of the set $S(q_1, q_2, \ldots, q_s)$ is an arbitrary function $t : \{q_1, q_2, \ldots, q_s\} \to \{0, 1\}$.

Observe that if t is a valuation sequence of some formula ϕ and t' is an extension of t over a finite subset of $O \cup S_0$ then t' is also a valuation sequence of ϕ.

It follows from Proposition 2 that all valuations which are extensions of a valuation sequence $t : \{q_1, q_2, \ldots, q_s\} \to \{0, 1\}$ are equal on the set $S(q_1, q_2, \ldots, q_s)$. This observation permits us to denote $V_t(\phi) = V(\phi)$ where t is a valuation sequence of ϕ and the valuation V is an arbitrary extension of t.

DEFINITION. Let ϕ be a formula and let t be an arbitrary valuation sequence of ϕ. We say that ϕ is *satisfied on* t is $V_t(\phi) = 1$.

PROPOSITION 3. *Let ϕ be a formula and let t, t' be valuation sequences of ϕ such that t' extends t. Then ϕ is satisfied on t if and only if ϕ is satisfied on t'.*

DEFINITION. (a) A formula is said to be *consistent* if there exists a valuation sequence of the formula which satisfies the formula.

(b) A set of formulas is said to be *consistent* if there exists a valuation sequence of this set which satisfies every formula from the set.

2. The quarrel theorem

Let
$$\mathcal{K} = \{a \lhd T(b) : a, b \in O\} \cup \{a \lhd L(b) : a, b \in O\}$$
Let Ω be an arbitrary finite subset of $\mathcal{K}$. In this section we shall formulate a necessary and sufficient condition of consistency of Ω. First let us introduce some definitions and notation.

DEFINITION. For a given Ω we define the relation $\sim_\Omega$ on the set O as follows:

$$a \sim_\Omega b \equiv (\{a \lhd T(b), a \lhd L(b), b \lhd T(a), b \lhd (L(a)\} \cap \Omega \neq \emptyset).$$

Let $O(\Omega) = \{a \in O : \ (\exists_{b \in O})(a \sim_\Omega b)\}$.

DEFINITION. A set $\mathcal{C} \subseteq \Omega$ is called a *chain* if there exists a one-to-one sequence $(c_i)_{i=1}^n$ of elements of $O(\Omega)$ such that $|\mathcal{C}| = n-1$ and the following condition is satisfied

$$a \sim_{\mathcal{C}} b \text{ iff } ((\exists_i \in \{1, \ldots, n-1\})(a, b \in \{c_i, c_{i+1}\}) \wedge (a \neq b)).$$

It follows from the definition that $O(\mathcal{C}) = \{c_i : i = 1, \ldots, n\}$. If for a chain $\mathcal{C}$ we have $a = c_1$ and $b = c_n$, then we write $\mathcal{C} = \mathcal{C}_{ab}$.

DEFINITION. A set $\mathcal{P}$ is called a *circle* if there exists a one-to-one sequence $(c_i)_{i=1}^n$ of elements of $O(\Omega)$ such that $|\mathcal{P}| = n$ and the following condition is satisfied

$$a \sim_{\mathcal{P}} b \text{ iff } (((\exists_i \in \{1, \ldots, n-1\})(a, b \in \{c_i, c_{i+1}\}) \wedge (a \neq b))$$
$$\vee (a = c_1 \wedge b = c_n) \vee (a = c_n \wedge b = c_1))$$

Obviously we have $O(\mathcal{P}) = \{c_i : \ i = 1, \ldots, n\}$.

DEFINITION. Every formula of the form $a \lhd T(b)$ is called a *T-formula*. Every formula of the form $a \lhd L(b)$ is called an *L-formula*.

Now we list a few simple lemmas.

LEMMA 1. *If Ω is consistent and $\Omega' \subseteq \Omega$, then Ω' is consistent.*

LEMMA 2. *Let $\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_n$, and let the sets $O(\Omega_1), O(\Omega_2), \ldots, O(\Omega_n)$ be pairwise disjoint, then:*

(a) $\Omega_1, \Omega_2, \ldots, \Omega_n$ *are pairwise disjoint*

(b) Ω *is consistent iff each* Ω_i, *for* $i = 1, \ldots, n$ *is consistent.*

Lemma 3. *Let* Ω *be consistent and let* $a, b \in O(\Omega)$ *be different elements such that there is no chain* $\mathcal{C}_{ab} \subseteq \Omega$. *Then there exist valuation sequences* $t_1, t_2 : O(\Omega) \to \{0, 1\}$ *such that* $t_1(a) = t_1(b)$ *and* $t_2(a) \neq t_2(b)$, *and* t_1, t_2 *both satisfy* Ω.

Lemma 4. *Let* Ω *be consistent and let* $a, b \in O(\Omega)$ *and* $c \notin O(\Omega)$. *Then:*

(a) Both the sets $\Omega \cup \{c \lhd T(a)\}$ *and* $\Omega \cup \{c \lhd L(a)\}$ *are consistent and for every valuation sequence* $t : O(\Omega) \cup \{c\} \to \{0, 1\}$ *we have:*

(i) $t(a) = t(c)$ *if* t *satisies the former set*

(ii) $t(a) \neq t(c)$ *if* t *satisfies the latter one;*

(b) $\Omega \cup \{c \lhd T(a)\}$ *is consistent if and only if there is a valuation sequence* t *satisfying* Ω *and such that* $t(a) = t(b)$.

Before we consider the general case, let us characterize consistency of chains and circles.

Theorem 1. *(a) Every is consistent.*

(b) Let t *be a valuation sequence which satisfies a chain* $\mathcal{C}_{ab}$. *Then* $t(a) = t(b)$ *if and only if the number of all* L*-formulas in* $\mathcal{C}_{ab}$ *is even.*

Proof. (a) Suppose that there are inconsistent chains, and let $\mathcal{C}_{ab}$ be such a chain with a minimal number of elements. Since for different a, c both the formulas $a \lhd T(c), a \lhd L(c)$ are consistent, so $|\mathcal{C}_{ac}| \geq 2$ and $\mathcal{C}_{ac}$ can be represented as the union of some chain $\mathcal{C}_{ab}$ and a singleton containing a T-formula or an L-formula. We may assume that $\mathcal{C}_{ac} = \mathcal{C}_{ab} \cup \{c \lhd T(b)\}$ or $\mathcal{C}_{ac} = \mathcal{C}_{ab} \cup \{c \lhd L(b)\}$. Since $|\mathcal{C}_{ab}| < |\mathcal{C}_{ac}|$, so $\mathcal{C}_{ab}$ is consistent. A contradiction. This completes the proof.

(b) (the proof by induction on $n = |\mathcal{C}_{ab}|$)

1) The case $n = 1$ is trivial.

2) Assume that the theorem holds for every chain $\mathcal{C}_{ab}$ with $|\mathcal{C}_{ab}| = n$. Let $\mathcal{C}_{ac}$ be an arbitrary chain with $|\mathcal{C}_{ac}| = n + 1$. We may assume that $\mathcal{C}_{ac}$ is one of the following unions:

(1) $\mathcal{C}_{ac} = \mathcal{C}_{ab} \cup \{c \lhd T(b)\}$

or

(2) $\mathcal{C}_{ac} = \mathcal{C}_{ab} \cup \{c \lhd L(b)\}$.

Let t be an arbitrary valuation sequence satisfying $\mathcal{C}_{ac}$. According to the form of the union, t satisfies one of the following conditions:

(1) $t(b) = t(c)$

or

(2) $t(b) \neq t(c)$. Using our inductive assumption, we obtain the following equivalences:

(1) $t(a) = t(c)$ if and only if the number of all L-formulas in $\mathcal{C}_{ab}$ is even.

(2) $t(a) \neq t(c)$ if and only if the number of all L-formulas in $\mathcal{C}_{ab}$ is odd.

Hence finally, $t(a) = t(c)$ if and only if the number of all L-formulas in $\mathcal{C}_{ac}$ is even. $\square$

As an easy consequence of Theorem 1 and Lemma 4, we obtain the following theorem.

THEOREM 2. *A circle $\mathcal{P}$ is consistent if and only if the number of all L-formulas in $\mathcal{P}$ is even.*

THEOREM 3. (the quarrel theorem) *A finite set $\Omega \subseteq \mathcal{K}$ is consistent if and only if the number of all L-formulas in each circle contained in Ω is even.*

PROOF. Necessity follows from Lemma 1 and Theorem 2. To prove sufficiency, assume that there is inconsistent subset of $\mathcal{K}$ such that the number of all L-formulas in each circle contained in it is even. Let Ω be such a set with the minimal number of elements. It follows from inconsistency of Ω that there are three different elements $a, b, c \in O(\Omega)$ such that $c \sim_\Omega a$ and $c \sim_\Omega b$. Thus, we may assume that one of the following cases holds:

(1) $\{c \lhd T(a), c \lhd T(b)\} \subseteq \Omega$

(2) $\{c \lhd L(a), c \lhd L(b)\} \subseteq \Omega$

(3) $\{c \lhd T(a), c \lhd L(b)\} \subseteq \Omega$

For each case we define the set Ω' as follows:

(1) $\Omega' = (\Omega \backslash \{c \lhd T(a), c \lhd T(b)\}) \cup \{a \lhd T(b)\}$

(2) $\Omega' = (\Omega \backslash \{c \lhd L(a), c \lhd L(b)\}) \cup \{a \lhd T(b)\}$

(3) $\Omega' = (\Omega \backslash \{c \lhd T(a), c \lhd L(b)\}) \cup \{a \lhd L(b)\}$

Since in each case the number of all L-formulas in every circle contained in Ω' remains even and $|\Omega'| < |\Omega|$, so Ω' is consistent.

In each case we consider three subcases:

(i) $c \notin O(\Omega')$

(ii) $c \in O(\Omega')$ and Ω' contains no chain $\mathcal{C}_{ac}$ and no chain $\mathcal{C}_{bc}$

(iii) $c \in O(\Omega')$ and Ω' contains a chain $\mathcal{C}_{ac}$.

Using Lemmas 1, 4 in the subcase (i), Lemmas 1, 3, 4 in the subcase (ii) and Lemmas 1, 4 with Theorems 1, 2 in the subcase (iii), we arrive at the conclusion that Ω is consistent which contradicts our assumption and completes the proof. Below we give details only for the case 3-(iii). The proofs of remaining cases are similar.

Proof of 3-(iii): Since $\mathcal{C}_{ac} \subseteq \Omega'$, then also $\mathcal{C}_{ac} \subseteq \Omega$. Hence the circle $\mathcal{P} = \mathcal{C}_{ac} \cup \{c \lhd T(a)\}$ is contained in Ω, and so the number of all L formulas in $\mathcal{P}$ is even. Consequently, the number of all L-formulas in $\mathcal{C}_{ac}$ is even, and for every valuation sequence t which satisfies Ω' (and so $\mathcal{C}_{ac} \subseteq \Omega'$) we have, by Theorem 1, $t(a) = t(c)$. Since $(a \lhd L(b)) \in \Omega'$, so we have also $t(a) \neq t(b)$ and $t(b) \neq t(c)$. Hence, by Lemma 4, $\Omega' \cup \{c \lhd T(a), c \lhd L(b)\}$ is consistent and so its subset Ω.

References

[1] Raymond Smullyan, **The Lady and the tiger**, New York, 1982.
[2] Raymond Smullyan, **Alice in Puzzle-Land**, New York, 1982.

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