Bulletin of the **S**ection of **L**ogic
Volume 19/4 (1990), pp. 147–150

reedition 2005 [original edition, pp. 147–151]

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INTUITIONISTIC SENTENTIAL CALCULUS WITH CLASSICAL IDENTITY

The aim of the present note is to show how the semantics for $ISCI$ given in [1] may be adapted for the case of intuitionistic sentential logic with the classical connective of identity.

1. Semantics

Where $(X, \leq)$ is a nonempty partially ordered set and, for each $x \in X$, $\mathcal{A}_x = \{A_x, -, \cap, \cup, \dot{\Rightarrow}, \dot{\Leftrightarrow}, \circ\}$ an algebra similar to the SCI-language $\mathcal{L} = (L, \neg, \wedge, \vee, \Rightarrow, \Leftrightarrow, \equiv), \emptyset \neq D_x \subset A_x$, we consider the class of matrices:

$$M_x = \{(\mathcal{A}_x, D_x) : x \in X\}.$$

We assume that $Av_{CI}(M_x)$ is the set of all $h \in \Pi\{Hom(\mathcal{L}, \mathcal{A}_x) : x \in X\}$ which satisfies the conditions:

(i) if $\Pi_x(h)(\alpha) \in D_x$, then $\forall_{y \geq x} \Pi_y(h)(\alpha) \in D_y$,

(1) $-\Pi_x(h)(\alpha) \in D_x$ iff $\forall_{y \geq x} \Pi_y(h)(\alpha) \notin D_y$,

(2) $\Pi_x(h)(\alpha) \cap \Pi_x(h)(\beta) \in D_x$ iff $\Pi_x(h)(\alpha) \in D_x$ and $\Pi_x(h)(\beta) \in D_x$,

(3) $\Pi_x(h)(\alpha) \cup \Pi_x(h)(\beta) \in D_x$ iff $\Pi_x(h)(\alpha) \in D_x$ or $\Pi_x(h)(\beta) \in D_x$,

(4) $\Pi_x(h)(\alpha) \dot{\Rightarrow} \Pi_x(h)(\beta) \in D_x$, iff $\forall_{y \geq x}(\Pi_y(h)(\alpha) \notin D_y$ or $\Pi_y(h)(\beta) \in D_y)$,

(5) $\Pi_x(h)(\alpha) \dot{\Leftrightarrow} \Pi_x(h)(\beta) \in D_x$ iff $\forall_{y \geq x}((\Pi_y(h)(\alpha) \in D_y$ and $\Pi_y(h)(\beta) \in D_y)$ or $(\Pi_y(h)(\alpha) \notin D_y$ and $\Pi_y(h)(\beta) \notin D_y))$,

(6') $\Pi_x(h)(\alpha) \circ \Pi_x(h)(\beta) \in D_x$ iff $\forall_{y \in X}\ \Pi_x(h)(\alpha) = \Pi_y(h)(\beta)$,

and additionally if α is an identity (i.e. the main connective of α is "$\equiv$")

(i') $\Pi_x(h)(\alpha) \in D_x$ iff $\forall_{y \in X} \Pi_y(h)(\alpha) \in D_y$

($\Pi_x(h)$ is the x-coordinate of $h \in \Pi Hom\{\mathcal{L}, \mathcal{A}_x) : x \in X\}$).

COROLLARY. *According to (i'), for every α being an identity, the conditions (1) and (4) reduce at every $x \in X$ to their classical counterparts:*
(1') $-\Pi_x(h)(\alpha) \in D_x$ iff $\Pi_x(h)(\alpha) \notin D_x$,
(4') $\Pi_x(h)(\alpha) \dot{\Rightarrow} \Pi_x(h)(\beta) \in D_x$ iff $(\Pi_x(h)(\alpha) \notin D_x$ or $\Pi_x(h)(\beta) \in D_x)$.
The same applies to (5) when α, β appearing there are identities.

With each M_X as above we associate the operation $C^{CI}_{M_X}$ of "matrix-like" consequence (cf. [1] p. 95).

DEFINITION. Let $\alpha \in L, B \subseteq L$. $\alpha \in C^{CI}_{M_X}(B)$ iff $\forall_{h \in Av_{CI}(M_X)} \forall_{x \in X}$ (if $\Pi_x(h)(B) \subseteq D_x$, then $\Pi_x(h)(\alpha) \in D_x$).

2. Axiom set and completeness

As the axiom set for $ISCI_{CI}$, i.e. the intuitionistic sentential calculus with the classical identity, we take the axiom set for $ISCI$ ([1]) enlarged with all formulas

EM $$\alpha \vee \neg\alpha$$

where α is identity.

We define $ISCI_{CI}$-consequence operation. $\alpha \in C_{CI}(B)$ iff α is derivable from the axiom set for $ISCI_{CI}$ and set B through the Detachment Rule.

COMPLETENESS THEOREM. *For any $\alpha \in L$ and $B \subseteq L$ the following conditions are equivalent:*
(*) $\alpha \in C_{CI}(B)$
(**) for each class of matrices M_X $\alpha \in C^{CI}_{M_X}(B)$.

PROOF. (an outline of (*) $\Rightarrow$ (**)). Assume that for some $\alpha \in L$ and $B \subseteq L$ $\alpha \notin C_{CI}(B)$. Then, by Lindenbaum's Lemma there exists a relatively maximal $ISCI_{CI}$-theory T_0 such that $C_{CI}(B) \subseteq T_0$ and $\alpha \notin T_0$. We take the class K of all relatively maximal $ISCI_{CI}$ theories which have the same set of identities as T_0. Subsequently, for every $T \in K$ we define $\sim_T$ putting

$$\alpha \sim_T \beta \text{ iff } \alpha \equiv \beta \in T,$$

for $\alpha, \beta \in L$, and consider

$$M_K = \{(\mathcal{L}/{\sim_T}, T/{\sim_T}) : T \in K\}.$$

Let $h \in \Pi\{Hom(\mathcal{L}, T/{\sim_T}) : T \in K\}$ be defined as follows:

$$\forall_{T \in K} \forall_{\alpha \in L} \Pi_T(h)(\alpha) = [\alpha]_T.$$

Thus, for any $ISCI_{CI}$-theory $T \in K$ $\Pi_T(h) : \mathcal{L} \to \mathcal{L}/{\sim_T}$ is a canonical homomorphism.

To show that $h \in Av_{CI}(M_K)$ we must check that for any $\alpha, \beta \in L$ the conditions (i), (1)-(5), (6') are satisfied; it follows from the construction of class M_K that if α is identity, the condition (i') is satisfied. Accordingly, it is nontrivial to demonstrate that for α-non-identity (1), (4) hold.

Condition (1). Suppose that $-\Pi_T(h)(\alpha) \in T/{\sim_T}$ and that for some $T' \in K, T \subseteq T', \Pi_{T'}(h)(\alpha) \in T'/{\sim_{T'}}$. Then $\neg\alpha \in T$. Since $\alpha \in T'$ and $\neg\alpha \in T'$, then $T' = L$.

Now assume that for any $T' \supseteq T, (T, T' \in K)$ $\Pi_{T'}(h)(\alpha) \notin T'/{\sim_{T'}}$ i.e. $\alpha \notin T'$. Since $\alpha, \neg\alpha \notin T$, then $C_{CI}(T, \alpha)$ is a consistent $ISCI_{CI}$-theory. Now, we may show that $C_{CI}(T, \alpha)$ has the same set of identities as T (T is a relatively maximal and for any identity $e, (e \vee \neg e) \in T$). So, there exists relatively maximal $ISCI_{CI}$-theory $T'' \in K$ and $C_{CI}(T, \alpha) \subseteq T''$. A contradiction.

Condition (4). Suppose that $\Pi_T(h)(\alpha) \dot{\Rightarrow} \Pi_T(h)(\beta) \in T/{\sim_T}$ and for some $T' \in K, T \subseteq T', \Pi_{T'}(h)(\alpha) \in T'/{\sim_{T'}}$, and $\Pi_{T'}(h)(\beta) \notin T'/{\sim_{T'}}$. Since $(\alpha \Rightarrow \beta), \alpha \in T'$, so $\beta \in T'$. A contradiction. Now assume that for any $T' \supseteq T, (T, T' \in K)$ $\Pi_{T'}(h)(\alpha) \notin T'/{\sim_{T'}}$, or $\Pi_{T'}(h)(\beta) \in T'/{\sim_{T'}}$. Moreover, let $\Pi_T(h)(\alpha) \dot{\Rightarrow} \Pi_T(h)(\beta) \notin T/{\sim_T}$. So $(\alpha \Rightarrow \beta), \alpha \notin T$, and $\beta \in T$. If $\neg\alpha \in T$ or $\beta \in T$, then since $((\neg\alpha \vee \beta) \Rightarrow (\alpha \Rightarrow \beta)) \in T$ $(\alpha \Rightarrow \beta) \in T$. So, assume that $\neg\alpha \notin T$ and $\beta \notin T$, then $C_{CI}(T, \alpha)$ is a consistent $ISCI_{CI}$-theory having the same set of identities as a member of K. Thus, there exists T'', $ISCI_{CI}$-theory maximal relatively to β such that $T'' \in K$ and $C_{CI}(T, \alpha) \subseteq T''$. A contradiction.

Since $C(B) \subseteq T_0, [\beta]_{T_0} \in T_0/{\sim_{T_0}}$ for any $\beta \in B$. Hence, $\Pi_{T_0}(h)(B) \subseteq T_0/{\sim_{T_0}}$. Simultaneously, since $\alpha \notin T_0, [\alpha]_{T_0} = \Pi_{T_0}(h)(\alpha) \notin T_0/{\sim_{T_0}}$. So, there exist $h \in Av_{CI}(M_K)$ and an $ISCI_{CI}$-theory $T_0 \in K$ such that:

$$\Pi_{T_0}(h)(B) \subseteq T_0/{\sim_{T_0}} \text{ and } \Pi_{T_0}(h)(\alpha) \notin T_0/{\sim_{T_0}} .$$

We have shown that $\alpha \notin C^{CI}_{M_K}(B)$. □

3. Remarks

3.1. It is straightforward that similarly the classical sentential calculus with intuitionistic identity, SCI_{II} may be constructed. To this aim, axiom set for $ISCI$ should be enlarged with all formulas $(\alpha \vee \neg\alpha)$ where α is non-identity). The definition of operation $C^{II}_{M_X}$ of an appropriate "matrix-like" consequence is based on the set of all valuations $h \in \Pi\{Hom(\mathcal{L}, A_x) : x \in X\}$ which satisfy conditions (i), (1)-(5),

(6) $\Pi_x(h)(\alpha) \circ \Pi_x(h)(\beta) \in D_x$, iff $\Pi_x(h)(\alpha) = \Pi_x(h)(\beta)$,

and (i') for α being non-identity.

3.2. The semantics for $ISCI$ presented in [1] (p. 94, 95) can be adjusted for SCI (i.e. the classical sentential calculus with classical identity). This would be the case if we require X to be a one-element set. Another way to do the same with the semantics for $ISCI_{CI}$ given in Section 1 is replacing (i) with (i'). Then the conditions (1), (4) and (5) become "classical" and (6) will be equivalent to (6').

References

[1] P. Łukowski, *Intuitionistic sentential calculus with identity*, **Bulletin of the Section of Logic** vol. 19 no. 3 (1990).

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