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Nguyen Cat Ho

GENERALIZED ALGEBRAS OF POST AND THEIR APPLICATIONS TO MANY-VALUED LOGICS WITH INFINITELY LONG FORMULAS

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I. Generalizations of Post algebras

Definition of Post Algebra of the Type A. Let $A = \langle A, \cap \rangle$ be a semilattice with zero 0. Let P be a distributive lattice with zero 0 and unit element 1. Let B be the set of all Boolean elements in P. P will be called generalized Post algebra of the type A provided that the following conditions are satisfied:

($P1$) There exists a subset $E = \{e_\alpha : \alpha \in A\} \cup \{e_v\}$ in P such that $e_0 = 0$, $e_v = 1$ and, for every α and β in A, if $\alpha \leqslant \beta$, then $e_\alpha \leqslant e_\beta$.

($P2$) For every element p in P there exists a decreasing sequence $\langle p_\alpha : \alpha \in A^{-1} \rangle$ of the type $A^{-1} = A - \{o\}$ of the elements in B, such that

(1) $p = \bigcup_{A^{-1}}(p_\alpha \cap e_\alpha)$ (monotonic representation of the element p).

($P3$) For every subset $A_0 \subset A$, there exists the sum $\bigcup_{\alpha \leqslant A_0} e_\alpha$ in P. Let us symbolize the sum $\bigcup_{\beta < \alpha} e_\beta$ by $e_{\underline{\alpha}}$.

($P4$) For every Boolean element b, if $b \cap e_\beta \leqslant e_\alpha$, where $\beta \nleqslant \alpha$, or $b \cap e_\alpha \leqslant e_{\underline{\alpha}}$, then $b = 0$.

Generalized Post algebra of type A with the distinguished elements e_α in E will be denoted by $P = \langle P; E; R\rangle$. From the definition accepted here it may be proved that for every element p there exists exactly one monotonic representation of this element. Hence it follows that for every α in E we may define operation D such that $D_\alpha p = p_\alpha$, $p \in P$.

This fact suggests that we many treat Post algebra of the type A as a lattice with the operations $D_\alpha, \alpha \in A^{-1}$:
(2) $\langle P; E; D_\alpha : \alpha \in A^{-1}\rangle$.

THEOREM. *Lattice (2) is a Post algebra of the type A if and only if the following conditions are satisfied:*

(i) $D_\alpha(p \cap q) = D_\alpha p \cap D_\alpha q$; $D_\alpha(p \cup q) = D_\alpha p \cup D_\alpha q$
(ii) $D_\alpha D_\beta p = D_\beta p$
(iii) $D_\alpha p - D_\beta p = 1$
(iv) $D_\alpha e_\beta = \begin{cases} 1 & \textit{if} \quad \alpha \leqslant \beta \\ 0 & \textit{if} \quad \alpha \not\leqslant \beta; \end{cases}$ $D_\beta e_\alpha = \begin{cases} 1 & \textit{if} \quad \beta < \alpha \\ 0 & \textit{if} \quad \beta \not< \alpha \end{cases}$
(v) For every p in P, $p = \bigcup_{\alpha \in A^{-1}} D_\alpha p \cap c_\alpha$.

If Post algebra (2) is a pseudo-Boolean lattice, then

$$D_\alpha(p \Rightarrow q) = \bigcap_{\beta \leqslant \alpha, \beta \notin Lim} (D\beta p \Rightarrow q)$$

and

$$D(-p) = -\bigcup_{\beta \leqslant \alpha, \beta \in Lim} D\beta p.$$

Definitions of subalgebras, homomorphisms and ideals for generalized Post algebras given by T. Traczyk and Ph. Dwinger appeared not suitable for the investigations into the problems concerning $\underline{m}$-representability of Post algebras of the type A. That is why these concepts were replaced by the ones which are more natural as well as more general. In particular, instead of the concept of ideal the concept of D-filter is utilized. This allows for a shift of almost all results on Post algebras of infinite order to the case of generalized Post algebras of the type A. Moreover, a theorem on $\underline{m}$-representability may be proved and Rasiowa-Sikorski's lemma for Boolean algebras may be generalized for the case of Post algebras. In a theorem on $\underline{m}$-representability contained in the complete version of this paper nine equivalent conditions of $\underline{m}$-representability were given for generalized Post algebras of the type A, where $\overline{\overline{A}} < m$ or A has the greatest element. The families

$$(R) \begin{cases} \bigcup_{j\in J_i^1} p_{ij} = p_i, & i \in I_1, \quad \overline{\overline{I}}_1 \text{ and } \overline{\overline{J}}_i^1 < m \\ \bigcap_{j\in J_i^2} q_{ij} = q_i, & i \in I_2, \quad \overline{\overline{I}}_2 \text{ and } \overline{\overline{I}}_i^2 < m \\ r_i \Rightarrow s_i = u_i, & i \in I, \quad \overline{\overline{I}} < m \end{cases}$$

where p, q, s, r, u with indices are in P, will be called $\underline{m}$-families.

The generalized lemma of Rasiowa-Sikorski will have now the following form:

THEOREM. *For every generalized Post algebra of the type A if $\overline{\overline{A}} < m$ or A has the greatest element, then the condition*

(*) "for every $\underline{m}$-family (R) and for every element $p \neq 1$ there exists a D_v-prime filter ∇ of the type A in P such that $p \notin \nabla$ and ∇ preserves all the operations in the family (R)"

is sufficient for $\underline{m}$-representability of the algebra P. If $\overline{\overline{A}} < m$, then the condition (*) is necessary as well.

II. Deductive systems $S(L_m^v, \Sigma)$ based on v-valued language L_m^v

Let m be an arbitrary regular cardinal. Let v be an ordinal such that $v < m$ or $v \notin Lim$ (v not being a limit ordinal).

Consider the language L_m^v whose alphabet consists of the elements of the following disjoint sets of symbols: 1^o) infinite set μ of sentential variables; 2^o) the set $\varepsilon_v = \{E_0, E_1, \ldots, E_v\}$ of sentential constants; 3^o) the set of sentential connectives: $\overline{D}_\alpha$, where $\alpha \leqslant v$, $\alpha \notin Lim$, $\rightharpoondown, \rightarrow$ and infinite connectives $\vee, \Lambda$; 4^o) the set of auxiliary symbols: (,).

Any sequence $\langle G_0, G_1, \ldots, G_i, \ldots \rangle i < 1$ of the symbols in $L)m^v$ will be called the expression of the length l of this language, where $0 < l < m$.

The set of formulas of the language L_m^V is the smallest set F_m^v of expressions of any length $0 < l < m$, which contains the union of the sets μ^m and ε_v and which is closed under $\overline{D}_\alpha, \rightharpoondown, \rightarrow, \vee, \Lambda$.

The results of C. Karp, G. Rousseau and H. Rasiowa suggest that the following logical axioms be accepted:

Group A

(a_4) $((\Lambda_{i<l}(F_i \to F_l) \to ((V_{i<l}F_i) \to F_l))$
(a_5) $((\Lambda_{i<l}(F_l \to F_i) \to ((F_l \to (\Lambda_{i<l}F_i)))$
(a_6) $(F_i' \to (V_{i<l}F_i))$
(a_7) $((\Lambda_{i<l}F_i) \to F_i')$; where $i' < l$.

The logical axioms in the group A are analogous to those of intuitionistic sentential logic given in the book "The mathematics of metamathematics" by Rasiowa and Sikorski.

Group B

(b_1) $\overline{D}_\alpha E_\beta, \beta \geqslant \alpha, \alpha \notin Lim$
(b_2) $\neg \overline{D}_\alpha E_\beta, \beta < \alpha, \alpha \notin Lim$
(b_3) $(\overline{D}_1 F \vee \neg \overline{D}_1 F)$
(b_4) $(\overline{D}_\alpha(V_{i<n}F_i) \leftrightarrow (V_{i<n}\overline{D}_\alpha F_i))$
(b_5) $(\overline{D}_\alpha(\Lambda_{i<n}F_i) \leftrightarrow (\Lambda_{i<n}\overline{D}_\alpha F_i))$
(b_6) $(\overline{D}_\alpha(F_0 \to F_1) \to (\overline{D}_\beta F_0 \to \overline{D}_\beta F_1))$; $\beta \leqslant \alpha$
(b_7) $(\overline{D}_\alpha \overline{D}_\beta F \leftrightarrow D_\beta F)$
(b_8) $(\overline{D}_\alpha \neg F \leftrightarrow \overline{D}_1 F)$
(b_9) $(F \leftrightarrow (V_{\alpha \leqslant v, \alpha \notin Lim}(\overline{D}_\alpha F \wedge E_\alpha)))$; for $m > v$
(b_9') $((\overline{D}_\alpha F \wedge E_\alpha) \to F); \alpha \leqslant v$, for $v \geqslant m$

We accept the following rules of deduction:
Rule of detachment: $F, F \to G/G$
Rule for conjunction: $F_i, i < l < m/\Lambda_{i<l}F_i$
Rule $RD : F/\overline{D}_v^o F$
where $\overline{D}_v^0 F$ denotes the formula $\overline{D}_v F$ if $v \notin Lim$ or $(\Lambda_{\alpha \leqslant v, \alpha \notin Lim}\overline{D}_\alpha F)$ if v is a limit number.

For $v \geqslant m$ we assume one more rule:

(R_v) $((\overline{D}_\alpha F \wedge E_\alpha) \to G)$; $\alpha \leqslant v$, $\alpha \notin Lim/F \to G$

The deductive system with the axioms of group A and B and with the above rules will be symbolized by $S(L_m^v)$. We may now define in a familiar way the concept of proof of the formula F on the grounds of the set Γ of formulas in $S(L_m^v)$ and we write $\vdash_\Gamma F$.

Generally, we consider the deductive system $S(L_m^v, \Sigma)$, where the set Σ of formulas is treated as a set of specific axioms.

Like in the case of language L_ω^2 of the classical sentential calculus, we may interpret the language L_m^v in v-element Post algebra $\varepsilon_v = \{e_\alpha : \alpha \leqslant v\}$ which is an $\underline{m}$-complete pseudo-Boolean algebra. We may interpret the language L_m^v in $\underline{m}$-complete Post algebras p_v of the type v which are, at the same time, pseudo-Boolean lattices. Then we may define the concepts of semantic validity, i.e. validity in ε_v (validity in P_v), semantic satisfiability (satisfiability in P_v) of the set Γ of formulas and semantic entailment on the grounds of the set Γ of formulas (which we shall write down as $\Vdash_\Gamma F$).

We have two definitions of completeness:

1^o) System $S(L_m^v, \Sigma)$ is said to be complete provided that for every formula $F, \vdash_F$ if and only if $\vdash_\Sigma F$,

2^o) System

We introduce the following equivalence in $F_m^v : F \sim_\Gamma F'$ if and only if $\vdash_\Gamma F \leftrightarrow F'$ in $S(L_m^v, \Sigma)$. Then the set $P(S, \Gamma)$ of all equivalence classes of formulas constitutes an $\underline{m}$-complete Post algebra of the type v which is pdeudo-Boolean. Such algebra is called Post-Lindenbaum algebra.

Definition. Double indexed sequence of formulas

(*C*) $\langle F_{ij} : i < l, j < l \rangle$; $0 < l, i < m$ will be said to be inconsistent (inconsistent for the formula F_0) provided that each of its subsequences that has at least one expression in common with every line of the matrix F_{ij}, contains some pair F and $\neg F$ (contains some pair F and $\neg F$ or F_0).

The theorems to follow establish the relationships between the concepts of completeness, strong completeness and representability of the respective Lindenbaum-Post algebras.

Theorem (on the conditions of completeness): *Let Σ be a set of semantically valid formulas in L_m^v. The following conditions are equivalent:*

(i) System $S = S(L_m^v, \Sigma)$ is complete.

(ii) For every sequence (C) inconsistent for F, the condition $\vdash E_1 \to (V_{j<l} F)ij)$ in S, for $i < l$, entails that $\vdash E_1 \to F$.

(iii) For every inconsistent sequence (C), the condition $\vdash E_1 \to V_{j<l} F_{ij}$ in S, for $0 < i < l$, entails that $\vdash \neg (V_{j<l} P_{oj})$.

(iv) Lindenbaum-Post algebra $P(S; \Gamma)$ is $\underline{m}$-representable.

(v) *If $\underline{m}$-complete Post algebra P_v is generated by the set G of the cardinality $n = \overline{\mu}$ and every formula in Σ is valid in P_v, then P_v is $\underline{m}$-representable.*

(vi) *If Γ is consistent with respect to S (i.e., if $S(L^v_m, \Sigma \cup \Gamma)$ is consistent) and $\overline{\overline{\Gamma}} < m$, then Γ is satisfiable.*

Theorem (on the conditions of strong completeness). *Assuming the same as in the previous theorem the following conditions are equivalent:*

(i) *System $S = S(L^v_m, \Sigma)$ is strongly complete.*

(ii) *Every set Γ of formulas, consistent with respect to S is satisfiable.*

(iii) *If Γ is consistent with respect to S, then Lindenbaum-Post algebra $P(S;\Gamma)$ is isomorphic with $\underline{m}$-complete Post field of the type v.*

(iv) *If P_v is $\underline{m}$-complete Post algebra of the type v generated by the set G of the cardinality $n = \overline{\overline{\mu}}$ and every formula in Σ is valid in P_v, then P_v is isomorphic with $\underline{m}$-complete Post field of the type $\vee$.*

Symbolize by $\pi_{\underline{m}}$ the set of all formulas of the form

(π_1) $E_1 \to (V_{i<l}(\Lambda_{j<l} F_{ij}))$, $0 < l < m$

where $\langle F_{ij} : i < l, j < l\rangle$ is an inconsistent sequence, and symbolize by $\delta_{\underline{m}}$ the set of all formulas of the form

(δ_1) $(\Lambda_{i<l}(V_{j<l} F_{ij})) \to (V_{g \in l} l(\Lambda_{i<l} F_{i\ g(i)}))$

for an arbitrary sequence $\langle F_{ij} : i < l, j < l\rangle, 0 < l < m$.

The subsequent theorems follow from the above two theorems:

Theorem. *The following systems are complete:*

1^o) *$S(L^v_{\omega_1})$ and $S(L^v_\omega)$, for $v < \omega_1$.*

2^o) *$S(L^v_m, \pi_{\underline{m}})$, where $v < m$.*

3^o) *$S(L^v_m, \delta_{\underline{m}})$, where $v < m$ and m is a strongly inaccessible cardinal.*

For the strong completeness we have:

Theorem. *System $S(L^v_m, \delta_n)$ is strongly complete provided that $v \leqslant n = \overline{\overline{\mu}}$ and $2^n < m$.*

Department of Mathematics and
Mechanics University of Warsaw