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A UNIFORM METHOD OF PROOF OF THE COMPLETENESS THEOREM FOR THE EQUIVALENTIAL PROPOSITIONAL CALCULUS AND FOR SOME OF ITS EXTENSIONS

The complete text of this paper will appear as [8].

1. Introduction

The present abstract is concerned with the adjustment of the method of proving the completeness of the classical propositional calculus which was described by G. Asser in [1], for the completeness proof of the equivalential fragment of this calculus.

The method described by G. Asser amounts in fact to a slight modification of the method of J. Łoś (cf. [4]) which is in turn an adaptation of the well-known L. Henkin's method of proving the completeness of the classical first order predicate calculus. Asser's method is based upon Tarski-Herbrand's deduction theorem for the implication connective. In the present abstract we utilize the deduction theorem for the equivalence connective stated and proved in [7].

In respect of the number of the theorems of the propositional calculus in question used in the proof of the completeness of this calculus the presented method seems to be the simplest of the hitherto known methods of proving the completeness of the purely equivalential propositional calculus and of some of its extensions (cf. [2], [3], [5] and [6]).

We denote the propositional variables by the letters: $p, q, r, \ldots$. The connectives of equivalence, negation and non-equivalence are denoted, respectively, by E, N, E'. Arbitrary formulas construed of the formulas x, y and of the connectives are symbolized by Exy, Nx, $E'xy$.

2. The completeness proof for the equivalential propositional calculus

In the present section we shall confine ourselves to considering only the propositional formulas construed exclusively of propositional variables and of the equivalence connective E.

The set of all the theorems of the equivalential fragment of the classical propositional calculus is defined now to be the smallest set comprising the formulas

$(A1)$ $EExzEyxEzy$
and
$(A2)$ $EExEyzEExyz$

which is closed under the rule defined by the following scheme

$(R1)$ $Exy, x/y$

which is called the rule of detachment.

By the rule $(R1)$ we deduce from axioms $(A1)$ and $(A2)$ some theorems which we shall utilize in the sequel.

$(T1)$ $EEyzEzy$
$(T2)$ $EyExExy$
$(T3)$ Ezz
$(T4)$ $EEyzEExzEyx$
$(T5)$ $EExzEEyxEyz$
$(T6)$ $EEyxEEyzExz$
$(T7)$ $EExEyzEEuyExEuz$
$(T6)$ $EExEyzEyExz$

Lemma 1. *For arbitrary propositional formulas x, y and for an arbitrary set of formulas X, if x can be deduced from the set $X + \{y\}$ by means of the rule $(R1)$ and axioms $(A1)$ and $(A2)$, then either x or Eyx can be deduced from X by means of $(R1)$, $(A1)$ and $(A2)$.*

Let x be an arbitrary formula. Let $\mathcal{L}(x)$ symbolize the family of the sets of formulas of the form $L(x)$ such that

(i) $L(x)$ is an extension of the set of all the theorems of the equivalential propositional calculus, closed under $(R1)$

(ii) $x \notin L(x)$

and

(iii) if $y \notin L(x)$, then x can be deduced from $L(x) + \{y\}$ by means of $(R1)$, $(A1)$ and $(A2)$, for an arbitrary formula y.

Lemma 2. *For an arbitrary formula x, either x is a theorem of the equivalential propositional calculus or the family $\mathcal{L}(x)$ is non-empty.*

Lemma 3. *For arbitrary formulas x, y, z and for an arbitrary $L(x) \in \mathcal{L}(x)$ the following conditions are equivalent:*

(a) $Eyz \in L(x)$

and

(b) $y \in L(x)$ *if and only if* $z \in L(x)$.

Proof from the Theorems $(T1)$, $(T2)$, $(T4)$ and Lemma 1.

Lemma 4. *Every tautology of the equivalential propositional calculus is a theorem of this calculus.*

Proof from Lemmas 2 and 3.

3. The completeness proof for the propositional calculus based on the equivalence and negation connectives

The set of all theorems of the propositional calculus with equivalence and negation is defined as the smallest set including $(A1)$, $(A2)$ and

$(A3)$ $\quad EExyENxNy$

and closed under $(R1)$.

From axioms $(A1)$, $(A2)$ and $(A3)$ we deuce, i.e., the subsequent theorems making use of $(R1)$.

$(T9)$ $\quad ENyEyENxx$

$(T10)$ $\quad EEyzENyxENxx$

We shall modify the definition of the family $\mathcal{L}(x)$ of the previous section as follows. Let x be an arbitrary formula. Let us symbolize by $\mathcal{L}_N(x)$ the family of the sets of formulas of the form $L(x)$, and such that

(i) $L(x)$ is an extension of the set of all theorems of the propositional calculus with the equivalence and negation connectives and closed under $(R1)$,
(ii) $x \notin L(x)$
(iii) if $y \notin L(x)$, then x can be deduced from $L(x)+\{y\}$ by means of $(R1)$, $(A1)$, $(A2)$ and $(A3)$, for an arbitrary formula y and
(iv) $Nx \in L(x)$.

Lemma 5. *For an arbitrary formula x, either x is a theorem of the propositional calculus with the equivalence and negation connectives or the family $\mathcal{L}_N(x)$ is non-empty.*

The proof of this lemma is analogous to that of Lemma 2.

Lemma 6. *For any formulas x, y, z and for a arbitrary $L(x) \in \mathcal{L}_N(x)$*
(a) $Ny \in L(x)$ if and only if $y \notin L(x)$.

Proof from Lemmas 1 and 3, and from Theorems $(T9)$ and $(T10)$.

Lemma 7. *Every tautology of the propositional calculus with equivalence and negation connectives is a theorem of this calculus.*

Proof from Lemmas 5 and 6.

4. The completeness proof for the propositional calculus based on the equivalence and non-equivalence connectives

The set of all theorems of the propositional calculus with the equivalence and non-equivalence is defined to be the smallest set including $(A1)$, $(A2)$ and

$(A4)$ $EExyEEzuEE'xzE'yu$
and closed under $(R1)$.
$(T11)$ $EE'xyEExyE'zz$
$(T12)$ $EE'xyEExEE'zzy$
$(T13)$ $ExEEyzEEE'zzzE'xy$
$(T15)$ $ExEEyzEEE'zzzE'yx$

Let us modify now the definition of the family $\mathcal{L}(x)$ which was given in the first section. Let x be an arbitrary propositional formula. Let us symbolize by $\mathcal{L}_{E'}(x)$ the family of sets of formulas of the form $L(x)$ and such that

(i) $L(x)$ is an extension of the set of all theorems of the propositional calculus with the equivalence and non-equivalence connectives and closed under $(R1)$,

(ii) $x \notin L(x)$, and $E'xx \notin L(x)$

and

(iii) if $y \notin L(x)$, then x can be deduced from $L(x) + y$ by means of $(R1)$, $(A1)$, $(A2)$ and $(A4)$, for an arbitrary y.

LEMMA 8. *For any formula x, either x is a theorem of the propositional calcvulus with the equivalence and non-equivalence connectives, or the family $\mathcal{L}_{E'}(x)$ is non-empty.*

LEMMA 9. *For any formulas x, y, z and for any $L(x) \in \mathcal{L}_{E'}(x)$:*

(a) $E'yz \in L(x)$ if and only if $[y \in L(x)$ if and only if $z \notin L(x)]$.

Proof from Lemmas 1, 3, 6 and Theorems $(T11)$, $(T12)$, $(T13)$ and $(T14)$.

LEMMA 10. *Every tautology of the propositional calculus with the equivalence and non-equivalence is a theorem of this calculus.*

Proof from Lemmas 8 and 9.

5. A simple generalization of the described method

Let F_k be a k-ary connective which satisfies the axiom

(F_k) $\quad EEx_1y_1EEx_2y_2 \ldots EEx_ky_kEF_kx_1x_2 \ldots x_kF_ky_1y_2 \ldots y_k$

We define the concept of tautology of the propositional calculus with the equivalence and with the F_k connective. Viz., let f_k be a k-ary operation in the set $\{0, 1\}$ defined inductively as follows:

(i) $f_1(x_1) = 1 - x_1$,

(ii) $f_{k+1}(x_1, x_2, \ldots, x_{k+1}) = e(f_k(x_1, x_2, \ldots, x_k), x_{k+1})$

where $k \geqslant 1$, $x_1, x_2, \ldots, x_{k+1} \in \{0,1\}$ and where $e(x,y) = 1$ if and only if $x = y$. Let f be an arbitrary mapping of the propositional variables into the set $\{0,1\}$ and let h be an arbitrary mapping such that for any propositional variable p

(i) $h(p) = f(p)$

and for any formulas $x, y, x_1, x_2, \ldots, x_k$

(ii) $h(Exy) = e(h(x), h(y))$

and

(iii) $h(F_k x_1 x_2 \ldots x_k) = f_k(h(x_1), h(x_2), \ldots, h(x_k))$.

Then x is a tautology if and only if $h(x) = 1$.

It can be proved now that every theorem of the propositional calculus with the connectives E and F_k is a tautology of this calculus.

For any formula x we define the family $\mathcal{L}_{F_k}(x)$ of the sets of the form $L(x)$, as follows:

(i) $L(x)$ is ana extension of the set of all theorems of the propositional calculus with the connectives E and F_k, and closed under $(R1)$,
(ii) $x \notin L(x)$,
(iii) if k is an even number, then $F_k \underbrace{xx\ldots x}_{k\ times} \notin L(x)$,
(iv) if k is an odd number, then $F_k \underbrace{xx\ldots x}_{k\ times} \in L(x)$

and

(v) if $y \notin L(x)$, then x can be deduced from $L(x)+\{y\}$ by means of $(R1)$, $(A1)$, $(A2)$ and (F_k), for an arbitrary y.

It can be proved that for any x, either x is a theorem of the propositional calculus with the connectives E and F_k, or the family $\mathcal{L}_{F_k}(x)$ is non-empty.

Furthermore, it can be demonstrated that for arbitrary $x_1, x_2, \ldots, x_k$ and for an arbitrary $L(x) \in \mathcal{L}_{F_k}(x)$:

(i) if $k = 1$, then $F_k x_1 x_2 \ldots x_k = F_1 x_1$ and in this case $F_1 x_1 \in L(x)$ if and only if $x \notin L(x)$,
(ii) if $k > 1$, then $F_k x_1 x_2 \ldots x_k \in L(x)$ if and only if $[F_{k-1} x_1 x_2 \ldots x_{k_1} \in L(x)$ if and only if $x_k \in L(x)]$.

Finally, it can be proved that every tautology of the propositional calculus with the connectives E and F_k is a theorem of this calculus.

References

[1] G. Asser, **Einfrühung in die matematische Logik**, Leipzig 1959.

[2] S. Leśniewski, *Grundzüge eines neuen Systems der Grundlagen der Mathematik. Einleitung und § 1–11*, **Fundamenta Mathematicae** 14 (1929), pp. 1–81.

[3] S. Leśniewski, *Bemerkungen zur Fortsetzung meiner Mittellung u.d.t. "Grundlagen eines neuen Systems der Grundlagen der Mathematik"*, **Collectanea Logica** 1 (1939), pp. 1–60, (Warszawa).

[4] J. Łoś, *An algebraic proof of completeness for the two-valued propositional calculus*, **Colloquium Mathematicum** 2 (1951), pp. 236–240.

[5] E. Ch. Mihailescu, *Recherches sur un sous-systeme du calcul de propositions*, **Annales Scientifiques de l'Universite' de Jassy. Partie I**, 23 (1937), pp. 106–124.

[6] H. Rasiowa, *Axiomatisation d'un syséme partiel de la théorie de la deduction*, **Comptes Rendus des seances de la Société des Sciences et des Lettres de Varsovie, Cl. III**, 40 (1947), pp. 22–37.

[7] S. J. Surma, *Method of natural deduction in equivalential and equivalential-negational propositional calculus*, **Universitas Iagellonica Acta Scientiarum Litterarumque, Schedae Logicae**, CCLXV, Fasc. VI, Kraków, 1971, pp. 55–68.

[8] S. J. Surma, *A survey of the results and methods of investigations of the equivalential propositional calculus*, [in:] **Studies in the History of Mathematical Logic**, Wrocław, Ossolineum, in print.

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