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AN ALGORITHM FOR FINDING FINITE AXIOMATIZATIONS OF FINITE INTERMEDIATE LOGICS

This paper was read to the Conference Logical Calculi, Warsaw, October 1971. It was also reported on November 10 and 17, 1971 at the seminar of the Department of Logic, Jagiellonian University, held by Professor Stanisław J. Surma in Cracow. Its full text will appear in Zeszyty Naukowe Uniwersytetu Jagiellońskiego, Prace z Logiki, Zeszyt 7.

In [7] the information was given that D. H. J. De Jongh has achieved in a constructive manner the result that every finite intermediate logic is finitely axiomatizable. The proof of this fact was published in [5] and credited by the author to A. S. Troelstra. The Troelstra's proof is also constructive, but however, it provides a rather cumbersome procedure of the axiomatization. In this paper a new constructive proof of the theorem of D. H. J. De Jongh is given. The proof suggests a comparatively feasible algorithm which enables us to find a finite axiom system for any finite intermediate logic.

The German capitals $\mathcal{A}, \mathcal{D}, \ldots$ will be used to denote pseudo-Boolean algebras called in the sequel – algebras – for the sake of simplicity. The corresponding Latin capitals $A, B, \ldots$ denote the domains of the algebras $\mathcal{A}, \mathcal{D}, \ldots$. The operations of join, meet, relative pseudo-complement, pseudo-complement are denoted by $\vee, \wedge, \rightarrow, \neg$ respectively. The unit element of the algebra $\mathcal{A}$ is denoted by $1_{\mathcal{A}}$. The symbol $\leqslant$ denotes an ordering. The smallest filter of $\mathcal{A}$ including the set $X \subseteq A$ is denoted by $[X)$. We will write $[x)$ instead of $[\{x\})$. The quotient algebra obtained by means of the congruence determined by the filter Ψ of is denoted by $\mathcal{A}/\Psi$. The symbol $[x]\Psi$ denotes the equivalence class of the element x. The following lemmas are simple corollaries of more general statements belonging to universal algebra (see [1] and [6]).

LEMMA I. *Let $\mathcal{A}$ be an algebra. Then the following condition hold:*

(i) $[\emptyset) = \{1_{\mathcal{A}}\}$*:*

(ii) if $X \neq \emptyset$, $X \subseteq A$ *then*
$[X) = \{y \mid x_1 \wedge \ldots \wedge x_n \leqslant y$ *for some* $x_1 \in X\}$.

LEMMA II. *Let $\mathcal{A}$ be an algebra, $x \in A$, $x \neq 1_{\mathcal{A}}$. Then there exists a filter Ψ of $\mathcal{A}$ such that $x \notin \Psi$ is a maximal filter with respect to this property.*

The following lemma is an algebraic analogon of the well-known deduction theorem.

LEMMA III. *Let $\mathcal{A}$ be ana algebra, $X \subseteq A$, $x \in A$, $y \in [X \cup \{x\})$. then $x \to y \in [X)$.*

An algebra $\mathcal{A}$ is said to be strongly compact iff there exists $x \in A$. such that

(i) $x \neq 1_{\mathcal{A}}$.

(ii) for every $y \in A$ if $y \neq 1_{\mathcal{A}}$ then $y \leqslant x$.

It is easy to see that if there exists an element such that (i) and (ii) then it is unique. Thus the symbol $*_{\mathcal{A}}$ will be used to denote such element in the strongly compact algebra $\mathcal{A}$.

LEMMA IV. *Let $\mathcal{A}$ be ana algebra, $x \in A$. Let Ψ be a filter of $\mathcal{A}$ such that $x \notin \Psi$ and Ψ is a maximal filter with respect to this property. Then $\mathcal{A}/\Psi$ is a strongly compact algebra.*

The Greek lowercase letters $\alpha, \beta, \ldots$ will be used to denote well-formed formulas built up by means of the variables $a, b, \ldots$ and the usual connectives $\vee, \wedge, \Rightarrow, \neg$.

We will write $\alpha \Leftrightarrow \beta$, $\bigvee(\alpha_i \mid i = 1, \ldots, n)$, $(\ldots(\alpha_1 \vee \alpha_2) \vee \ldots) \vee \alpha_n$, $(\ldots(\alpha_1 \wedge \alpha_2) \wedge \ldots) \wedge \alpha_n$ respectively. The symbol Cn is used to denote the consequence operation defined by means of the theorems of intuitionistic propositional calculus, the substitution rule, and the detachment rule. The content of the algebra $\mathcal{A}$ is denoted by $E(\mathcal{A})$. By the intermediate logic we mean $E(\mathcal{A})$ for any non-degenerate algebra $\mathcal{A}$. The script capitals $\mathcal{A}, \mathcal{B}, \ldots$ will be used to denote intermediate logics. If X is a set of formulas then the symbol $SC(X)$ denotes the set of all $E(\mathcal{A})$ such that $X \subseteq E(\mathcal{A})$ and $\mathcal{A}$ is a finitely generated strongly compact algebra. We will write $SC(\alpha)$ instead of $SC(\{\alpha\})$.

Lemma V. (A. V. Kuznecov, V. Ja. Gerciu [4]). *$Cn(X) = \bigcap(\mathcal{A} \mid \mathcal{A} \in SC(X))$ for every set of formulas X.*

Corollary. *$SC(X) = SC(Y)$ iff $Cn(X) = Cn(Y)$.*

If $\mathcal{A}$ is a finite strongly compact algebra then the formula

$$\begin{aligned}\chi_{\mathcal{A}} = \quad & (\bigwedge((a_x \vee a_y) \Leftrightarrow a_{x\vee y} \mid x,y \in A)\wedge \\ & \bigwedge((a_x \wedge a_y) \Leftrightarrow a_{x\wedge y} \mid x,y \in A)\wedge \\ & \bigwedge((a_x \Rightarrow a_y) \Leftrightarrow a_{x\to y} \mid x,y \in A)\wedge \\ & \bigwedge(\neg\, a_x \Leftrightarrow a_{\neg x} \mid x \in A))a_{*_{\mathcal{A}}}\end{aligned}$$

is called the characteristic formula of $\mathcal{A}$ [2].

Lemma VI. (V. A. Jankov [3]). *$\chi_{\mathcal{A}} \notin E(\mathcal{A})$ for every finite strongly compact algebra $\mathcal{A}$.*

Lemma VII. *Let $\mathcal{A}$ be a finite strongly compact algebra, let $\mathcal{D}$ be an algebra. If v is a refuting valuation of $\chi_{\mathcal{A}}$ on $\mathcal{D}$ then $v(a_x) \neq v(a_y)$ for every $x, y \in A$, $x \neq y$.*

Lemma VIII. (V. A. Jankov [3]). *Let $\mathcal{A}$ be a finite strongly compact algebra, let $\mathcal{D}$ be an algebra. Then $\chi_{\mathcal{A}} \notin E(\mathcal{D})$ iff $\mathcal{A}$ is embeddable in some quotient algebra of $\mathcal{D}$.*

Corollary. (V. A. Jankov [3]). *Let $\mathcal{A}$ be a finite strongly compact algebra, let $\mathcal{D}$ be an algebra. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is embeddable in some quotient algebra of $\mathcal{D}$;*

(ii) *$\chi_{\mathcal{A}} \notin E(\mathcal{D})$;*

(iii) *$E(\mathcal{D}) \subseteq E(\mathcal{D})$.*

Remark: It is obvious that $Cn(\chi_{\mathcal{A}}) = \bigcap(\mathcal{A} \mid \mathcal{A} \not\subseteq E(\mathcal{A}))$ since $\mathcal{A} \not\subseteq E(\mathcal{A})$ implies $Cn(\chi_{\mathcal{A}}) \subseteq \mathcal{A}$ and there is $Cn(\chi_{\mathcal{A}}) \not\subseteq E(\mathcal{A})$.

Let us define the formula γ_n, $n = 1, 2, \ldots$ as follows
$\gamma_n = \bigvee(a_i \Leftrightarrow a_j \mid i, j = 1, \ldots, n-1, i \neq j)$

Lemma IX. *Let $\mathcal{A}$ be an algebra. If $\mid A \mid \leqslant n$ then $\gamma_n \in E(\mathcal{A})$.*

LEMMA X. (C. G. McKay [5]) *Let $\mathcal{A}$ be a strongly compact algebra. If $\gamma_n \in E(\mathcal{A})$ then $\mid A \mid \leqslant n$.*

By the finite intermediate logic we mean $E(\mathcal{A})$ for any finite, non-degenerate algebra $\mathcal{A}$.

THEOREM. *Every finite intermediate logic is finitely axiomatizable.*

PROOF. Assume that $\mathcal{A}$ is an algebra and $\mid A \mid = n$. It is obvious that there is only a finite number (up to isomorphism) of the strongly compact algebras having at most n elements. Hence by X we get that the set $SC(\gamma_n) - SC(E(\mathcal{A}))$ is finite which implies that any logic belonging to $SC(\gamma_n) - SC(E(\mathcal{A}))$ is included in a maximal one.

Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be all the maximal logics in $SC(\gamma_n) - SC(E(\mathcal{A}))$. Thus for every $i = 1, \ldots, m$ there exists a strongly compact algebra $\mathcal{D}_i$ such that $\mathcal{D}_i = E(\mathcal{D}_i)$. Since by X every such algebra is finite then one can write out the characteristic formulas $\chi_{\mathcal{D}_1}, \ldots, \chi_{\mathcal{D}_m}$. We will prove that $X = \{\gamma_n, \chi_{\mathcal{D}_1}, \ldots, \chi_{\mathcal{D}_m}\}$ is an system of $E(\mathcal{A})$.

By the corollary to the lemma V it suffices to show that $SC(X) = SC(E(\mathcal{A}))$. Assume that $\mathcal{A} \in SC(E(\mathcal{A}))$. Then $\gamma_n \in \mathcal{A}$ by IX and it follows that

$$\mathcal{A} \not\subseteq E(\mathcal{D}_i), i = 1, \ldots, m.$$

Thus by the corollary to the Lemma VIII we have that $\chi_{\mathcal{A}_i} \in \mathcal{A}$, $i = 1, \ldots, m$ which gives that $X \subseteq \mathcal{A}$ and consequently $\mathcal{A} \in SC(X)$. To prove the converse inclusion let us assume that $\mathcal{A} \in SC(X)$ and $\mathcal{A} \notin SC(E(\mathcal{A}))$. From the definition of X it follows that $SC(X) \subseteq SC(\gamma_n)$. Thus $\mathcal{A} \in SC(\gamma_n) - SC(E(\mathcal{A}))$ and consequently $\mathcal{A} \subseteq E(\mathcal{D}_i)$ for some $i = 1, \ldots, m$.

Finally, applying the corollary to the Lemma VIII we get that $\chi_{\mathcal{D}_i} \notin \mathcal{A}$ which is a contradiction. Q.E.D.

The proof just given suggests the following simple procedure for finding an axiom system of $E(\mathcal{A})$, $\mid A \mid = n$.

(i) Find all (up to isomorphism) the strongly compact algebras having at most n elements.

(ii) Mark any algebra $\mathcal{D}$ such that $\chi_{\mathcal{D}} \notin E(\mathcal{A})$.

(iii) If $\mathcal{D}$, $\mathcal{L}$ are distinct algebras not marked yet and $\chi_{\mathcal{D}} \notin E(\mathcal{L})$ then mark $\mathcal{L}$.

(iv) Write out the axioms of $E(\mathcal{A})$ i.e. γ_n and all the characteristic formulas of the unmarked algebras.

Remark: By analyzing the proof of the theorem it is easy to notice that instead of $\{\gamma_n, \chi_{\mathcal{D}_i}, \ldots, \chi_{\mathcal{D}_m}\}$ as an axiom system of $E(\mathcal{A})$ one can take any set of formulas Y such that:

(i) $Y \subseteq E(\mathcal{A})$;
(ii) $\gamma_n \in Cn(Y)$;
(iii) for every $i = 1, \ldots, n$ there exists $\alpha \in Y$ such that $\alpha \notin E(\mathcal{D}_i)$.

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