Bulletin of the **S**ection of **L**ogic
Volume 20/1 (1991), pp. 2–6

reedition 2005 [original edition, pp. 2–6]

Branislav R. Boričić

INTERPOLATION THEOREM FOR INTUITIONISTIC $S4$*

Abstract

By using the Maehara method a proof that the intuitionistic analogue of the Lewis modal system $S4$ has both Craig and the Lyndon interpolation property is presented.

An intuitionistic version of the Lewis modal system $S4$ can be formulated as an extension of the Heyting propositional calculus H by adding the following axiom-schemata

T. $\Box A \to A$

4. $\Box A \to \Box\Box A$

K. $\Box(A \to B) \to (\Box A \to \Box B)$

and the necessitation rule

$$A/\Box A$$

This system, denoted by $HS4$, was considered many times in the context of the investigations of intuitionistic modal logics (see [2], [3], [4], [7] or [8]). H. Ono (see [7]) has been formulated a corresponding cut-free sequent calculus, say $GHS4$, for $HS4$ obtained as a modification of the sequent calculus for $S4$ introduced by M. Ohnishi and K. Matsumoto (see [6]) by adding the rules concerning the necessity operator to the propositional part of Gentzen's LJ:

---

*The result contained in this paper was presented at the Fourth Conference of the Yugoslav Seminar for Logic and Computer Science held in Dubrovnik, September 6-9, 1990. This work was supported in part by Fond za nauku Srbije.

$(\Box \vdash)\ \dfrac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}$

$(\vdash \Box)\ \dfrac{\Gamma \vdash A}{\Gamma \vdash \Box A}$

where Δ consists of the at most one formula in $(\Box \vdash)$, and Γ is a sequence of formulae of the form $\Box B$ in $(\vdash \Box)$.

In this note, by using the Maehara method (see [9]) a proof of the interpolation theorem for $HS4$ will be presented. More precisely, we shall show that the system $HS4$ has both the Craig and the Lyndon interpolation property.

In the usual way, by induction on the degree of modal formula A, we define its *positive* A^+ and *negative* A^- *parts*:

(i) if A is a propositional letter, then $A^+ = \{A\}$ and $A^- = \emptyset$;

(ii) if A is $B \wedge C$ or $B \vee C$, then $A^+ = \{A\} \cup B^+ \cup C^+$ and $A^- = B^- \cup C^-$;

(iii) if A is $B \to C$, then $A^+ = \{A\} \cup B^- \cup C^+$ and $A^- = B^+ \cup C^-$;

(iv) if A is $\neg B$, then $A^+ = \{A\} \cup B^-$ and $A^- = B^+$;

(v) if A is $\Box B$, then $A^+ = \{A\} \cup B^+$ and $A^- = B^-$.

By atA we denote the set of all propositional letters occurring in A, and $atA^+ =_{def} atA \cap A^+$ and $atA^- =_{def} atA \cap A^-$. These definitions can be extended to the sequence of formulae in a natural way.

A logic L will be said to have the *Lyndon interpolation property* iff from $\vdash_L A \to B$ it follows:

(1) $\vdash_L \neg A$

(2) $\vdash_L B$

or

(3) $(atA^+ \cap atB^+ \neq \emptyset$ or $atA^- \cap atB^- \neq \emptyset)$ and there exists a formula C such that $\vdash_L A \to C, \vdash_L C \to B$ and $(atC^+ \subseteq atA^+ \cap atB^+$ and $atC^- \subseteq atA^- \cap atB^-)$ (see [5]).

A logic L will be said to have the *Craig interpolation property* iff from $\vdash_L A \to B$ it follows:

(1) $\vdash_L \neg A$

(2) $\vdash_L B$

or

(3) $atA \cap atB \neq \emptyset$ and there exists a formula C such that $\vdash_L A \to C, \vdash_L C \to B$ and $atC \subseteq atA \cap atB$ (see [1]).

Obviously, if a logic L has the Lyndon interpolation property, then L has the Craig interpolation property as well.

As the immediate consequences of the lemma given below and the cut-elimination theorem for $GHS4$ (see [7]), we have the following statements:

THEOREM. *The logic $HS4$ has the Lyndon interpolation property.*

THEOREM. *The logic $HS4$ has the Craig interpolation property.*

LEMMA. *If a sequent $\Gamma \vdash \Delta$ is provable in $GHS4$ then, for each partition $(\Gamma_1; \Gamma_2)$ of the sequence Γ, one of the following two conditions is satisfied:*

(i) sequent $\Gamma_1 \vdash$ or $\Gamma_2 \vdash \Delta$ is provable in $GHS4$;

(ii) $at\Gamma_1^+ \cap (at\Gamma_2^- \cup at\Delta^+) = X \neq \emptyset$ or $at\Gamma_1^- \cap (at\Gamma_2^+ \cup at\Delta^-) = Y \neq \emptyset$, and there exists a formula C, such that the sequents $\Gamma_1 \vdash C$ and $\Gamma_2, C \vdash \Delta$ are provable in $GHS4, atC^+ \subseteq X$ and $atC^- \subseteq Y$.

In the case (3) of the above definitions, the formula C is called an *interpolant* (or the *Lyndon* and respectively *Craig interpolant*). Similarly, in case (iii) of our lemma, the formula C is an *interpolant* of the sequent $\Gamma \vdash \Delta$. The Maehara method, used in the following proof, describes a construction of the Lyndon interpolant.

PROOF. This lemma is provable by induction on the length of a cut-free proof for the sequent $\Gamma \vdash \Delta$ in $GHS4$. Although the cases concerning the propositional connectives are known, in order to justify the induction step of the proof, the rules for introducing implication and necessity operator will be discussed here.

Case ($\rightarrow\vdash$):

$$\frac{\Gamma \vdash A \qquad \Pi, B \vdash \Delta}{\Gamma, A \rightarrow B, \Pi \vdash \Delta}$$

The possible partitions of the antecedent part of the lower sequent are $(\Gamma_1, \Pi_1; \Gamma_2, A \rightarrow B, \Pi_2)$ and $(\Gamma_1, A \rightarrow B, \Pi_1; \Gamma_2, \Pi_2)$ where $(\Gamma_1; \Gamma_2)$ and $(\Pi_1; \Pi_2)$ are partitions of Γ and Π, respectively. In case of the first partition, by the induction hypothesis, the following subcases are possible:

(I) $\Gamma_1 \vdash$ or $\Pi_1 \vdash$ is provable;

(II) both $\Gamma_2 \vdash A$ and $\Pi_2, B \vdash \Delta$ are provable, or

(III) there are the interpolants C and D of the corresponding sequents.

Then, in (I), by some applications of structural rules, we infere the sequent $\Gamma_1, \Pi_1 \vdash$; if (II), by ($\rightarrow\vdash$), we have $\Gamma_2, A \rightarrow B, \Pi_2 \vdash \Delta$, and, finally, if (III), we can see that the corresponding interpolant will be the formula $C \wedge D$. The second partition can be considered in the same way, with the remark that in case we have interpolants C and D corresponding to the upper sequents of the rule, the interpolant of the lower sequent will be the formula $C \rightarrow D$.

Case ($\vdash\rightarrow$):

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$$

We consider the partition $(\Gamma_1; \Gamma_2)$ of the sequence Γ. By the induction hypothesis we have provability of $\Gamma_1 \vdash$ or $\Gamma_2, A \vdash B$ (wherefrom we infere $\Gamma_2 \vdash A \rightarrow B$), or the case with the interpolant: $\Gamma_1 \vdash C; \Gamma_2, C, A \vdash B$, wherefrom we infere $\Gamma_2, C \vdash A \rightarrow B$ and conclude that the same formula C is an interpolant of the lower sequent.

Case ($\Box \vdash$): we consider the following two partitions: $(\Gamma_1; \Box A, \Gamma_2)$, and $(\Gamma_1, \Box A; \Gamma_2)$. The subcases without the interpolant can be justified easily. Otherwise, the interpolant is the same formula being, by the induction hypothesis, the interpolant of the upper sequent.

Case ($\vdash \Box$): there is just the $(\Gamma_1; \Gamma_2)$-partition, which will be trivial in the subcases without the interpolant, but otherwise, if we have the interpolant C of the upper sequent, by the induction hypothesis, then, from $\Gamma_1 \vdash C$ and $\Gamma_2, C \vdash A$, by the rules ($\Box \vdash$) and ($\vdash \Box$), we infere $\Gamma_1 \vdash \Box C$ and $\Gamma_2, \Box C \vdash \Box A$, and conclude that the formula $\Box C$ will be an interpolant of the lower sequent. Q.E.D.

We emphasize there the interpolant constructed in the proof given above retains the character of Lyndon's interpolant.

Let us remark that the results presented in this paper can be extended to the case of $HS4$ with quantifiers, by using the same method.

References

[1] W. Craig, *Linear reasoning. A new form of the Herbrand-Gentzen theorem*, **The Journal of Symbolic Logic** vol. 22 (1957), pp. 250–268.

[2] K. Došen, *Models for stronger normal intuitionistic modal logic*, **Studia Logica** vol. 44 (1985), pp. 39–70.

[3] K. Došen, *Higher-level sequent systems for intuitionistic modal logic*, Publications de l'Institut Mathématique (Beograd) vol. 39(53 (1986), pp. 3–12.

[4] D. M. Gabbay, **Semantical Investigations in Heyting's Intuitionistic Logic**, D. Reidel, Dordrecht, 1981.

[5] R. C. Lyndon, *An interpolation theorem in the predicate calculus*, **Pacific Journal of Mathematics** vol. 9 (1959), pp. 129–142.

[6] M. Ohnishi, K. Matsumoto, *Gentzen method in modal calculi*, **Osaka Math. J.** vol. 9 (1957), pp. 113–130.

[7] H. Ono, *On some intuitionistic modal logics*, Publications RIMS, Kyoto University vol. 13 (1977), pp. 687–722.

[8] D. Prawitz, **Natural Deduction. A Proof-Theoretical Study**, Almqvist and Wiksell, Stockholm, 1965.

[9] G. Takeuti, **Proof Theory**, North-Holland Publ. Comp. Amsterdam, 1975 (2nd ed. 1987).

University of Belgrade
Kamenička 6
1100 Beograd
Yugoslavia