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DECISION PROBLEM FOR FINITE EQUIVALENTIAL ALGEBRAS

By an equivalential algebra we mean any groupoid $\mathcal{A} = (A, \leftrightarrow)$ which can be embedded into a $(\leftrightarrow)$-reduct of Brouwerian semilattice, where the operation $\leftrightarrow$ is determined by the term $x \leftrightarrow y = (x \to y) \wedge (y \to x)$. This notion was introduced by J. Kabziński and A. Wroński in [2]. They showed that the class $\mathbf{E}$ of all equivalential algebras is equationally definable by the following identities:

$$(x \leftrightarrow x) \leftrightarrow y = y,$$

$$((x \leftrightarrow y) \leftrightarrow z) \leftrightarrow z = (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow z),$$

$$((x \leftrightarrow y) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z)) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z) = x \leftrightarrow y,$$

The variety $\mathbf{E}$ is congruence permutable and

$$p(x, y, z) = ((x \leftrightarrow y) \leftrightarrow z) \leftrightarrow (((x \leftrightarrow z) \leftrightarrow z) \leftrightarrow x)$$

serves as a Malcev term. Since among the identities in the language $(\leftrightarrow)$ the same are satisfied by $\mathbf{E}$ and $\mathbf{BS}$ (i.e. variety of all Brouwerian semilattices) then any Malcev condition which holds in $\mathbf{E}$ holds also in $\mathbf{BS}$. However the converse fails to hold. The simplest, but important, example is congruence distributivity. To see this let us only note that among subvarieties of $\mathbf{E}$ there is the smallest non-trivial one, namely the variety $\mathbf{E}_2$ of Boolean groups which is generated by the two element group and can be axiomatized, relative to $\mathbf{E}$, by adjoining the associativity law. This immediately gives that there is no non-trivial congruence distributive variety of equivalential algebras. Nevertheless subvarieties of $\mathbf{E}$ form a distributive lattice. Moreover, if for a variety $\mathbf{V} \subseteq \mathbf{BS}$ we define $\mathbf{V}^e$ to be a class of all equivalential algebras satisfying all identities of $\mathbf{V}$ involving $\leftrightarrow$ as the only operation then we have $\mathbf{E} = \mathbf{BS}^e$.

Equivalential algebras are subreducts (i.e. subalgebras of reducts) of Brouwerian semilattices. Actually for any subvariety $\mathbf{V}$ of $\mathbf{BS}$ all algebras from $\mathbf{V}^e$ are subreducts of those from $\mathbf{V}$, i.e. $\mathbf{V}^e = S(\mathbf{V}^r)$, where $\mathbf{V}^r$ is a class of all reducts of algebras from $\mathbf{V}$. In particular we have $\mathbf{E}_2 = \mathbf{BS}_2^e = S(\mathbf{BS}_2^r)$, where $\mathbf{BS}_2$ is the smallest non-trivial subvariety of $\mathbf{BS}$. Actually we have even $\mathbf{E}_2 = \mathbf{BS}_2^r$. However it is the only non-trivial subvariety of $\mathbf{E}$ with this property. For this reason we consider decision problem separately for all equivalential algebras and for those being reducts of Brouwerian semilattices.

THEOREM 1. *For any subvariety* $\mathbf{V}$ *of* $\mathbf{E}$ *the following conditions are equivalent:*

(1) $Th(\mathbf{V}_{fin})$ *is decidable.*
(2) $Th(\mathbf{V}_{fin})$ *is not hereditarily undecidable.*
(3) $\mathbf{V} \subseteq \mathbf{E}_2$.

THEOREM 2. *For any subvariety* $\mathbf{V}$ *of* $\mathbf{BS}$ *the following conditions are equivalent:*

(1) $Th(\mathbf{V}^r_{fin})$ *is decidable.*
(2) $Th(\mathbf{V}^r_{fin})$ *is not hereditarily undecidable.*
(3) $\mathbf{V}$ *is generated by chains.*

From Theorem 1 we get that the only non-trivial variety of equivalential algebras with decidable first order theory is the variety $\mathbf{E}_2$ of all Boolean groups, a result which one can easily infer from the S. Burris and R. McKenzie [1] characterization of congruence modular varieties with decidable elementary theory.

References

[1] S. Burris and R. McKenzie, *Decidability and Boolean representations*, **Memoirs of the American Mathematical Society** no. 246, (1981).

[2] J. K. Kabziński and A. Wroński, *On equivalential algebras*, Proceedings of the 1975 International Symposium on Multiple-Valued Logic, Indiana University, Bloomington, May 13-16, 1975, pp. 419–428.

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