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POST RELATION ALGEBRAS

In Orłowska (1987, 1988, 1990a) a relationship has been established between various nonclassical logics and relation algebras. It has been shown that formulas of the logics can be treated as some relations and propositional operations as relational operations. That observation becomes an inspiration for development of nonclassical relation and cylindric algebras (Orłowska 1990b). In the present paper we introduce a class of nonclassical relation algebras which correspond to Post logics, and we propose a new method of algebraization of those logics.

1. ω^+-valued and m-valued Post logic

The language of ω^+-valued Post logic includes the symbols from the following pairwise disjoint sets:

$VARPROP$ an infinite denumerable set of propositional variables

$\{E_j\}_{0 \leq j \leq \omega}$ a chain of type ω^+ of propositional constants

$\{\neg, \vee, \wedge, \rightarrow\}$ the set of propositional operations of negation, disjunction, conjunction, and implication, respectively.

$\{D_i\}_{1 \leq i < \omega}$ a family of unary propositional operations

The set $FORP$ of formulas of the logic is the smallest set including sets $VARPROP$ and $\{E_j\}_{0 \leq j \leq \omega}$ and closed with respect to all the propositional operations.

Kripke style semantics for the logic has been developed in Maximowa and Vakarelov (1974). By a Post-Kripke frame we mean a relational system of the form

$$K = (W, R, \{d_i\}_{1 \le i < \omega})$$

which satisfies the conditions below:
$(P1)$ W is a nonempty set (of states), R is a nonempty binary relation in W (accessibility relation), each $d_i : W \to W$ is a function in W
$(P2)$ R is reflexive
$(P3)$ R is transitive
$(P4)$ $d_i d_j x = d_i x$
$(P5)$ $(x, d_1 x) \in R$
$(P6)$ $(d_{i+1} x, d_i x) \in R$ for any i
$(P7)$ $(x, y) \in R$ implies $(d_i x, d_i y) \in R$ for any i
$(P8)$ $(d_i x, x) \in R$ iff $(x, d_{i+1} x) \in R$
$(P9)$ For any $x \in W$ there is i such that $(d_i x, x) \in R$.

Models of the logic are systems $M = (K, m)$ where K is a Post-Kripke frame and $m : VARPROP \to P(W)$ is a meaning function which assigns sets of states to propositional variables and satisfies the condition:

(m) If $(s, t) \in R$ and $s \in m(p)$, then $t \in m(p)$ for $p \in VARPROP$.

As usual, we define satisfiability of formulas by states in a model:
M, s *sat* p iff $s \in m(p)$ for $p \in VARPROP$
M, s *sat* $A \vee B$ iff M, s *sat* A or M, s *sat* B
M, s *sat* $A \wedge B$ iff M, s *sat* A and M, s *sat* B
M, s *sat* $A \to B$ iff for all $t \in W$ if $(s, t) \in R$ and M, t *sat* A then M, t *sat* B
M, s *sat* $\neg A$ iff for all $t \in W$ if $(s, t) \in R$ then not M, t *sat* A
M, s *sat* $D_i A$ iff $M, d_i s$ *sat* A
M, s *sat* E_j iff $(d_j s, s) \in R,\ 0 < j < \omega$
M, s *sat* E_ω, not M, s *sat* E_0 for any $s \in W$.

A formula A is true in a model M iff M, s *sat* A for all $s \in W$. A formula is valid if it is true in all models.

As usual, $A \leftrightarrow B$ stands for $(A \to B) \wedge (B \to A)$.

Axiomatization of the logic has been given in Rasiowa (1973). The axiom and rules are listed below:

$A0$ Axioms of the intuitionistic propositional calculus
$A1$ $D_i(A \vee B) \leftrightarrow D_i A \vee D_i B$
$A2$ $D_i(A \wedge B) \leftrightarrow D_i A \wedge D_i B$

A3 $D_i(A \to B) \leftrightarrow (D_1 A \to D_1 B) \wedge \ldots \wedge (D_i A \to D_i B)$
A4 $D_i(\neg A) \leftrightarrow \neg D_1 A$
A5 $D_i D_j A \leftrightarrow D_j A$
A6 $D_i E_j$ for $i \leq j$, $\neg D_i E_j$ for $i > j$
A7 $D_{i+1} A \to D_i A$
A8 E_ω
A9 $D_1 A \vee \neg D_1 A$
A10 $D_i A \wedge E_i \to A$

The rules of inference are modus ponens and

$$(r1) \qquad \frac{A}{D_i A \ for\ any\ i}$$

$$(r2) \qquad \frac{D_i A \ for\ any\ i}{A}$$

Completeness of the given axiomatization with respect to an algebraic semantics has been proved in Rasiowa (1973), and completeness with respect to the class of all models based on Post-Kripke frames bas been given in Maximova and Vakarelov (1974).

Post logics of a finite order m include finitely many operations D_i for $1 \leq i \leq m-1$, and finitely many constants E_j for $0 \leq j \leq m-1$. The respective Kripke frames of order m are obtained from Post-Kripke frames defined above by restricting the number of operations d_i to $i = 1, \ldots, m-1$. Axiomatization of the logic can be obtained by replacing axioms *A*8 and *A*10 by the following:

A8′ E_{m-1}
A10′ $A \leftrightarrow E_1 \wedge D_1 A \vee \ldots \vee E_{m-1} \wedge D_{m-1} A$.

Rules of inference are modus ponens and $(r1)$ for $i = 1, \ldots, m-1$.

Post relation algebras

Let a Post-Kripke frame $K = (W, R, \{d_i\}_{1 \leq i < \omega})$ be given. Following a method developed by Orłowska (1990b) we define a class of relation algebras corresponding to the Post logic. The elements of each of those algebras are binary relation of the form $X \times W$ for $X \subseteq W$, that is they are sets (or equivalently unary relations) which are "dummy embedded" into $W \times W$. The operations in the algebras are counterparts of propositional operations in the Post logic. For the sake of simplicity they are denoted in the same way as the respective logical operators. Let $A, B \in \{X \times W : X \subseteq W\}$, the we define:

($p\cup$) $A \cup B = \{(x, y) : (x, y) \in A \text{ or } (x, y) \in B\}$
($p\cap$) $A \cap B = \{(x, y) : (x, y) \in A \text{ and } (x, y) \in B\}$
($p-$) $-A = \{(x, y) : \text{not } (x, y) \in A\}$
($p \rightarrow$) $A \rightarrow B = \{(x, y) : \text{for all } z \text{ if } (x, z) \in R \text{ and } (z, y) \in A \text{ then } (z, y) \in B\}$
($p\neg$) $\neg A = \{(x, y) : \text{for all } z \text{ if } (x, z) \in R \text{ then not } (z, y) \in A\}$
(pD) $D_i A = \{(x, y) : (d_i x, y) \in A\}$
(pE) $E_i = \{(x, y) : (d_i x, x) \in R\}$ for $0 < j < \omega, E_0 = \emptyset, E_\omega = W \times W$.

Clearly, $\cup, \cap, -$ are Boolean operations. E_j are constants or, equivalently, 0-ary relational operations. We also have:

$$(d) \qquad -D_i A = D_i - A.$$

Proposition 2.1.

($P10$) *If* $i \leq j$, *then* $(d_j x, d_i x) \in R$
($P11$) $(d_i x, y) \in R$ *implies* $(x, d_i y) \in R$
($P12$) $(d_i x, x) \in R$ *and* $(d_i x, y) \in A$ *imply* $(x, y) \in A$
($P13$) $(d_i x, x) \in R$ *and* $(x, y) \in A$ *imply* $(d_i x, y) \in A$

Proposition 2.2. *For any Post-Kripke frame K set $\{X \times W : X \subseteq W\}$ is closed with respect to the relational operations defined above.*

By a full Post relation algebra determined by frame K we mean an algebra

$$fullPoRA(K) = (\{X \times W : X \subseteq W\}, \cup, \cap, -, \rightarrow, \neg, \{D_i\}_{1 \leq i < \omega}, \{E_j\}_{0 \leq j \leq \omega})$$

Consider the class

$$fullPoRA = \{fullPoRA(K) : K \text{ is a Post-Kripke frame}\}.$$

Then the class of Post relation algebras is defined as

$$PoRA = SP(fullPoRA)$$

where S and P are the operation of taking isomorphic copies of subalgebras and direct products, respectively.

Consider an algebra of binary relations over the set W

$$fullRA(W) = (P(W \times W), \cup, \cap, -, 1, \circ, ^{-1}, I)$$

where $\cup, \cap, -$ are Boolean operations, $1 = W \times W, \circ$ is the relational composition, $^{-1}$ is the converse operation, and I is the identity relation. In a standard way (Nemeti 1990) we define the classes of set relation algebras:

$$fullRA = \{fullRA(W) : W \text{ is a set}\}$$

$$RA = SP(fullRA)$$

Post relation algebras are closely related to the ω^+-valued Post logic. Let $t' : VARPROP \to CON$ be a one to one mapping of the set of propositional variables onto the set of relational constants. We define a translation function t from formulas of the Post logic into relational terms:
$t(p) = t'(p)$ for $p \in VARPROP$, $t(E_j) = E_j$,
$t(\neg A) = \neg t(A)$, $t(D_i A) = D_i t(A)$,
$t(A \vee B) = t(A) \cup t(B)$, $t(A \wedge B) = t(A) \cap t(B)$, $t(A \to B) = t(A) \to t(B)$.

The respective definitions lead to the following theorem:

Proposition 2.3. *The following conditions are equivalent:*
(a) formula F is valid in ω^+-valued logic
(b) $t(F) = E_\omega$ holds in every algebra from $PoRA$.

We show that operations $\to, \neg, D_i$ for $1 \leq i < \omega$, and E_j for $0 \leq j \leq \omega$ as well as conditions (P2),...,(P8) are definable by means of relational terms over RA. Let $d_i \subseteq W \times W$, for $1 \leq i < \omega$ be binary relations in W such that for every i we have $d_i \cap D_i^{-1} \subseteq I$, that is they all are functions. They are intended to be a relational counterpart of function d_i in frame K, and for the sake of simplicity we denote then by the same symbols.

Proposition 2.4.
(a) $A \to B = -(R \circ (A \cap -B))$
(b) $\neg A = -(R \circ A)$
(c) $D_i A = d_i \circ A$
(d) $E_j = (I \cap (d_j \circ R)) \circ 1$ *for* $0 < j < \omega, E_0 = -1, E_\omega = 1.$

PROPOSITION 2.5.

(a) Condition $(P2)$ *holds in a frame* K *iff* $-I \cup R = 1$ *holds in* $fullRA$ *over set* W *of states of* K

(b) Condition $(P3)$ *holds iff* $-(R \circ R) \cup R = 1$

(c) Condition $(P4)$ *holds iff* $-I \cup d_j \circ d_i \circ I \circ d_i^{-1} = 1$

(d) Condition $(P5)$ *holds iff* $-I \cup d_1 \circ R^{-1} = 1$

(e) Condition $(P6)$ *holds iff* $-I \cup d_{i+1} \circ R \circ d_i^{-1} = 1$

(f) Condition $(P7)$ *holds iff* $-R \cup d_i \circ R \circ d_i^{-1} = 1$

(g) Condition $(P8)$ *holds iff* $-I \cup -(d_i \circ R) \cap -(d_{i+1} \circ -R^{-1}) \cup (d_i \circ R) \cap (d_{i+1} \circ -R^{-1}) = 1$

It seems that condition $(P9)$ is not expressible in a form of a relational equation. However, if we confine ourselves to algebras with finitely many operations D_i, then the respective class turns out to be a generalized reduct of the class of set relation algebras.

Let a Post-Kripke frame of order m be given. By a full Post relation algebra of order m over K we mean an algebra of the form

$$fullPo_mRA(K) = \\ = (\{X \times W : X \subseteq W\}, \cup, \cap, -, \rightarrow, \neg, D_1, \ldots, D_{m-1}, E_0, \ldots, E_{m-1})$$

where the operations $\cup, \cap, -$ are Boolean operations, $\rightarrow$ and $\neg$ are defined by $(p \rightarrow)$ and $(p\neg)$, respectively, D_i for $i = 1, \ldots, m-1$ and E_j for $j = 0, \ldots, m-1$ are defined by (pD) and (pE) respectively, with E_ω replaced by E_{m-1}. In a similar way we define classes of algebras:

$$fullPo_mRA = \{fullPo_mRA(K) : K \text{ is a Post-Kripke frame of order } m\}$$

$$Po_mRA = SP(fullPo_mRA)$$

Let $(P9')$ be the condition obtained from $(P9)$ by assuming that $i = 1, \ldots, m-1$.

PROPOSITION 2.6. *Condition* $(P9')$ *holds in Post-Kripke frame of order* m *iff* $-I \cup (d_1 \cup \ldots \cup d_{m-1}) \circ R = 1$ *holds in the algebra* $fullRA(W)$ *where* W *is a set of states of* K.

In view of propositions 2.2, 2.3 and 2.4 we conclude that any Post relation algebra of order m is a generalized reduct of a set relation algebra, namely we have:

Proposition 2.7.

$$Po_m RA = S(RdRA).$$

It is easy to see that Post relation algebras of order m correspond to the m-valued Post logic, namely we have:

Proposition 2.8. *The following conditions are equivalent:*
(a) A formula F is valid in m-valued Post logic
(b) $t(F) = E_{m-1}$ holds in every algebra from $Po_m RA$.

In a natural way we can obtain a set of equations defining the class of Post relation algebras of order m. Let $EQPo_m$ be the following set of equations derived from axioms of the m-valued Post logic.

($e1$) $A \cap (A \to B) = A \cap B$

($e2$) $(A \to B) \cap B = B$

($e3$) $(A \to B) \cap (A \to C) = (A \to (B \cap C))$

($e4$) $(A \to A) \cap B = B$

($e5$) $\neg(A \to A) \cup B = B$

($e6$) $A \to (\neg(A \to A)) = \neg A$

($e7$) $D_i(A \cup B) = D_i A \cup D_i B$

($e8$) $D_i(A \cap B) = D_i A \cap D_i B$

($e9$) $D_i(A \to B) = (D_1 A \to D_1 B) \cap \ldots \cap (D_i A \to D_i B)$

($e10$) $D_i(\neg A) = \neg D_1 A$

($e11$) $D_i D_j A = D_j A$

($e12$) $D_i E_j = E_{m-1} A$ for $i \leq j$, $D_i E_j = E_0$ for $i > j$

($e13$) $D_1 A \cup \neg D_1 A = E_{m-1}$

($e14$) $A = (E_1 \cap D_1 A) \cup \ldots \cup (E_{m-1} \cap D_{m-1} A)$

Proposition 2.9. *Class $Po_m RA$ is definable by $EQPo_m$.*

It would be interesting to investigate Post relation algebras within the relation-algebraic framework (Henkin, Monk and Tarski 1985, Nemeti 1990). In particular the algebras introduced in the present paper should be treated as abstract algebras, not necessarily set algebras, and a representation theory for them could be developed.

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