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AN OPEN PROBLEM IN TARSKI'S CALCULUS OF DEDUCTIVE SYSTEMS

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The notation and terminology of this paper follow [2], and are dual to those of [6] and [7]. If L is a language in the narrow sense, Cn may be any consequence operation on sets of sentences of L that includes classical sentential logic. Henceforth when we talk of the language L we intend to include reference to some fixed, though unspecified, operation Cn. $\mathbf{X}$ is a *deductive system* if $\mathbf{X} = Cn(\mathbf{X})$. Sentences x, z that are logically equivalent with respect to Cn – that is $x \in Cn(\{z\})$ and $z \in Cn(\{x\})$ – are identified. If $\mathbf{X}$ and $\mathbf{Z}$ are systems we often write $\mathbf{X} \vdash x$ instead of $x \in \mathbf{X}$ and $\mathbf{Z} \vdash \mathbf{X}$ instead of $\mathbf{X} \subseteq \mathbf{Z}$. If $\mathbf{X} = Cn(\{x\})$ for some sentence x, $\mathbf{X}$ is (*finitely*) *axiomatizable*. The set theoretical intersection of $\mathbf{X}$ and $\mathbf{Z}$ has the logical force of disjunction, and is written $\mathbf{X} \vee \mathbf{Z}$; $Cn(\mathbf{X} \cup \mathbf{Z})$, the smallest system to include both $\mathbf{X}$ and $\mathbf{Z}$, is written $\mathbf{XZ}$. If K is a family of systems, $\bigvee[K]$ and $\bigwedge[K]$ may be defined in an analogous way. The logically strongest system $\mathbf{S}$ is the set of all sentences of L; the weakest system $\mathbf{T}$ is defined as $Cn(\emptyset)$. The *autocomplement* $\mathbf{Z}'$ of $\mathbf{Z}$ is defined to be the strongest system to complement $\mathbf{T}$, namely the system $\bigwedge[\{\mathbf{Y} : \mathbf{Y} \vee \mathbf{Z} = \mathbf{T}\}]$. More generally we may define $\mathbf{X} - \mathbf{Z}$ as $\bigwedge[\{\mathbf{Y} : \mathbf{X} \vdash \mathbf{Y} \vee \mathbf{Z}\}]$. In terms of this operation to *remainder*, $\mathbf{Z}'$ is identical with $\mathbf{T} - \mathbf{Z}$. The class of all deductive systems forms a distributive lattice under the operations of concatenation and $\vee$; and indeed a Brouverian algebra (a relatively authocomplemented lattice with unit) under concatenation, $\vee$ – and $\mathbf{T}$.

In [6] Tarski established the principal results of the calculus of deductive systems. Theorem 17 shows that when non-axiomatizable systems exist in L - that is, when the calculus of systems goes genuinely beyond the calculus of sentences – the autocomplement does not obey all the laws of classical negation; in particular the law of non-contradiction, $\mathbf{XX}' = \mathbf{S}$, holds if and only if $\mathbf{X}$ is axiomatizable. Other classical laws can also fail: Theorems 23 and 26 deal respectively with the law of double negation $\mathbf{X} = \mathbf{X}''$ and the law $\mathbf{X}'\mathbf{X}'' = \mathbf{S}$. Systems obeying the first law may be called *regular*; those obeying the second Tarski called *convergent*. It is clear that only axiomatizable systems are both regular and convergent. At the end of [6] Tarski stated, but did not prove, that neither laws hold universally in a language in which non-axiomatizable systems are present. The existence of uncountably many divergent systems is stated explicitly in Theorem 38 of [7]. A simple example of a system that is not regular is any system Ω that is both unaxiomatizable and complete; that is, an unaxiomatizable system, not identical with $\mathbf{S}$, whose only proper extension is $\mathbf{S}$. It is easily shown that any such Ω is convergent; indeed, $\Omega' = \mathbf{T}$. These various results of Tarski's are recorded (and for the most part proved) in Theorems 1.6.1, 3.0, 3.1, 2.3.2, 3.3, and 1.6.2 of [2].

The main business of [2] was to investigate the possibility of irregular divergent systems, a question not addressed by Tarski. It was proved (Theorem 3.4) that there are no such systems in a language L in which there is only one unaxiomatizable complete Ω; but that there are uncountable many such when the number of unaxiomatizable complete Ω is at least two (Theorem 3.5.1). The question arises of whether this result concludes what may be said about different kinds of deductive systems; and if not, how it may be extended and generalized.

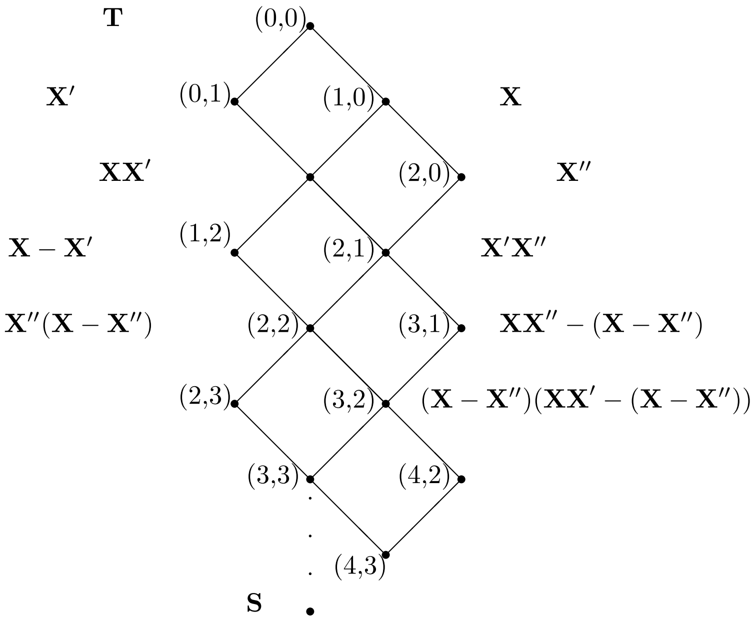

Figure 1

Figure 1 shows the free Brouwerian algebra on one free generator **X**. This is the dual of the free Heyting algebra described in Theorem 4.6 of [5] and in [4]; see also [1], pp. 182-185. (I am indebted to Professor Wroński for drawing may attention to [5].) Two related questions may be asked of the label # that a node bears:

(I) What conditions on **X** are necessary and sufficient for $\# = \mathbf{S}$?
(II) What conditions (if any) on the language L are necessary and sufficient for $\# = \mathbf{S}$ to hold for all systems **X** of L?

One set of answers to (I) in the segment form $(0,0)$ to $(2,2)$ is given in table 1 (the first two answers are not meant to be illuminating):

$(0,0)$	$\mathbf{T} = \mathbf{S}$	iff	$\mathbf{T} = \mathbf{S}$
$(1,0)$	$\mathbf{X} = \mathbf{S}$	iff	$\mathbf{X} = \mathbf{S}$
$(0,1)$	$\mathbf{X}' = \mathbf{S}$	iff	$\mathbf{X} = \mathbf{T}$
$(1,1)$	$\mathbf{X}'\mathbf{X}'' = \mathbf{S}$	iff	$\mathbf{X}$ is axiomatizable
$(2,0)$	$\mathbf{X}'' = \mathbf{S}$	iff	$\mathbf{X}' = \mathbf{T}$ [$\mathbf{T}$ is dense]
$(2,1)$	$\mathbf{X}'\mathbf{X}'' = \mathbf{S}$	iff	$\mathbf{X}'$ is axiomatizable [$\mathbf{X}$ is convergent]
$(1,2)$	$\mathbf{X} - \mathbf{X}'' = \mathbf{S}$	iff	$\mathbf{X} = \mathbf{X}''$ [$\mathbf{X}$ is regular]
$(2,2)$	$\mathbf{X}''(\mathbf{X} - \mathbf{X}'') = \mathbf{S}$	iff	$\mathbf{X}''$ finitely extends $\mathbf{X}$ [$\mathbf{X}'' = \mathbf{XZ}$ for some axiomatizable $\mathbf{Z}$]

Table 1

There is no special difficulty in extending such answers indefinitely far down the lattice. Table 2 takes some minor liberties with notation.

$(k+2,k) = \mathbf{S}$	iff	$(k,k) = (k,k+1)$
$(k+2,k+1) = \mathbf{S}$	iff	$(k,k+1)$ finitely extends (k,k)
$(k+1,k+2) = \mathbf{S}$	iff	$(k+1,k) = (k+2,k)$
$(k+2,k+2) = \mathbf{S}$	iff	$(k+2,k)$ finitely extends $(k+1,k)$

Table 2

Much more interesting and much more taxing, is question (II), though matters are simple enough at the beginning. To say that for some label # the identity $\# = \mathbf{S}$ is universally valid in L is to say that the lattice terminates at the node labeled #, which is the zero element. It follows that as none of the nodes $(k+2,k), (k+1,k+2)$, and $(k+2,k+1)$ can properly be the zero element, there is no language in which, for example, either $\mathbf{X} - \mathbf{X}'' = \mathbf{S}$ or $\mathbf{X}'\mathbf{X}'' = \mathbf{S}$ is universally valid unless $\mathbf{X}\mathbf{X}' = \mathbf{S}$ is too. Some of Tarski's results can thus be read off Figure 1 (it is not pretended that this amounts to a proof); in any language L in which there exist unaxiomatizable systems there will exist also systems that are irregular as well as systems that are divergent. Likewise $\mathbf{X}'' = \mathbf{S}$ is universally valid only if $\mathbf{X} = \mathbf{S}$ is universally valid; there will always exist systems that are not dense (for example the system $\mathbf{T}$) in a language L that contains more than a single system. Thus the only labels # such that $\# = \mathbf{S}$ holds universally, but for no lower label $\mathbf{Z}$ does $\mathbf{Z} = \mathbf{S}$ hold universally are those that label nodes of the form (k, k). The question is whether each such label is characteristic of some family of language; and if so, which.

Note that although no other labels can singly be characteristic of a family of language, two together can be. For example, as noted above, a necessary and sufficient condition for every system of L to be either convergent or regular, and for not all systems to be both, is that L contain exactly one unaxiomatizable complete system; such languages are therefore characterized by the pair of nodes (1,2) and (2,1). For the segment from (0,0) to (2,2), Table 3 sums up what is known. It is assumed that the identity or disjunction of identities in question must hold for every system $\mathbf{X}$ of the language.

Examples of each possibility may easily be provided. If the classical consequence operation is extended so much that every sentence follows from every other, and from none, we obtain a language in which $\mathbf{S}$ is the only deductive system; plainly there is no complete system here, and $\mathbf{T} = \mathbf{S}$. If L contains only two sentences, the inconsistency s and the tautology $\neg s$, then $\mathbf{T} = Cn(\neg s)$ is the only complete system; if $\mathbf{X} \neq \mathbf{S}$ then $\mathbf{X} = \mathbf{T}$, so $\mathbf{X} = \mathbf{S}'$. Sentential calculus with finitely many primitives yields only finitely many complete systems, all of them axiomatizable. An example in which there is exactly one unaxiomatizable complete system is given by elementary logic with identity = as the only predicate symbol. Here there are countably many axiomatizable complete systems, each of which states the finite cardinality of the universe;

$(0,0))$	$\mathbf{T} = \mathbf{S}$ iff L contains no complete systems
$(0,1)/(1,0))$	$\mathbf{X} = \mathbf{S}$ or $\mathbf{X}' = \mathbf{S}$ iff L contains at most one complete system
$(1,1))$	$\mathbf{X}\mathbf{X}' = \mathbf{S}$ iff L contains at most finitely many complete systems *equivalently* iff L contains no unaxiomatizable complete systems
$(1,2)/(2,1))$	$\mathbf{X}'\mathbf{X}'' = \mathbf{S}$ or $\mathbf{X} - \mathbf{X}'' = \mathbf{S}$ iff L contains at most finitely many unaxiomatizable complete systems

Table 3

the conjunction of all their negations, which intuitively asserts that the universe is infinite, is readily shown to be unaxiomatizable and complete (see [2], Theorem 2.3). In section 5 of [7] Tarski gives an example of a language in which there are two unaxiomatizable complete systems. A rather simple example is obtaining by adding to the language of elementary logic with identity a monadic predicate symbol P, then postulating as a logical axiom the sentence

$$\forall_a Pa \vee \forall_a \neg Pa.$$

The only two unaxiomatizable complete systems in this language both assert that the universe is infinite: one says in addition that all elements satisfy P, the other that all elements satisfy $\neg P$. It is not entirely obvious, but may be shown, that if $\mathbf{X}$ is any system in this language then $\mathbf{X}''$ is a finite extension of $\mathbf{X}$.

The open question in my title is this: What is the next step? How do we generalize these results through the lattice displayed in Figure 1? In particular, what should the next term be in the contradicting sequence *complete system*, *unaxiomatizable complete system*,...? Can the family of languages that are characterized by the node (2,2) be described as those in which there are no unaxiomatizable complete systems of some more specific kind?

Let Σ be the class of all complete systems of some language L. In Lemma 7 of [3] Mostowski showed that if $\mathcal{M}^d$ is the derived set of a set $\mathcal{M} \subseteq \Sigma$ (under a familiar topology due to Stone), then the second element of the contradicting sequence $\Sigma, \Sigma^d, \Sigma^{dd}$,... is the class of unaxiomatizable complete systems. This sequence is identical with the following, where Ω is a variable ranging over Σ.

$$\Sigma(0) \quad = \Sigma$$

$$\Sigma(k+1) \quad = \{\Omega : \Omega \text{ extends } \bigvee[\Sigma(k)], \text{ but not finitely}\}$$

Since $\bigvee[\Sigma] = \mathbf{T}$, the class $\Sigma(1)$ is just the class of unaxiomatizable Ω. If this is identical with Σ, then $\Sigma(k) = \Sigma$ for all k; if however, it is non-empty but finite then each element of it finitely extends $\bigvee[\Sigma(1)]$. Thus $\Sigma(2)$ and all succeeding $\Sigma(k)$ are empty. If going one step further, $\Sigma(1)$ is infinite then, it may be shown, there is a complete system that is an extension, but not a finite extension, of $\bigvee[\Sigma(1)]$. And so on.

I conjecture that the node (k, k) is the zero element of the lattice of Figure 1 if and only if $\Sigma(k)$ is empty. The conjecture holds for $k = 0, 1$, and 2 and seems reasonable enough. At present a proof is waiting.

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