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André Fuhrmann

CONDITIONAL LOGICS AND CUMULATIVE LOGICS

Introduction

Inference ivolving (so-called counterfactual) conditionals is a well-understood variety of nonmonotonic modes of reasoning. There is, accordingly, some prospect that by establishing links between logics of conditionals and varieties of nonmonotonic reasoning our understanding of the latter will improve. The purpose of this paper is to exhibit a close relationship between familiar conditional logics and a kind of nonmonotonic reasoning recently studied under the name of cumulative inference by Kraus, Lehmann and Magidor (henceforward KLM). I refer the reader to KLM [1] for information concerning the origins and motivation of systems of cumulative inference.

KLM claim (in [1, 170]) that the link between their work and previous work on conditional logic "is mainly at the level of the formal systems and not at the semantic level". And indeed, the semantics of KLM does not seem to fit easily into the range of semantics offered for conditionals (see [2] or [3] for a survey). But while cumulative inference *can* be modeled in a way that contrasts with familiar semantics for conditionals, it need not be thus modeled. The link between cumulative and conditional logic is just as strong at the level of semantics as it is at the level of proof theory.

Systems of cumulative inference

Cumulative logics are based on a sentential language $\mathcal{L}$ with $\wedge$ (conjunction) and $\neg$ (negation) chosen as primitive sentential connectives. There are two relations of inference between formulae: the straight turnstile, $\vdash$, denoting a familiar notion of deducibility, and a curly turnstile $\mid\!\sim$, used for a nonmonotonic relation inference (cumulative inference). The following system **CB** emerges as a basis for all logics discussed in the context of cumulative inference; it is based on the classical sentential calculus and is closed under the rule of Modus Ponens.

System **CB** ("basic" cumulative logic)

$$\frac{A \dashv\vdash B \quad B \mid\!\sim C}{A \mid\!\sim C} \qquad \text{Left Logical Equivalence}$$

$$\frac{A \mid\!\sim B \; B \vdash C}{A \mid\!\sim C} \qquad \text{Right Weakening}$$

$$\frac{A \mid\!\sim B \; A \mid\!\sim C}{A \mid\!\sim B \wedge C} \qquad \text{And}$$

$$A \mid\!\sim \top \qquad \text{Truth}$$

Semantics for stronger systems – as those proposed by KLM – will be forthcoming by constraining classes of models in a manner corresponding to the postulates added to **CB**.

Background: Frames for conditionals

The language $\mathcal{L}^{\leadsto}$ to be modelled extends the boolean language $\mathcal{L}$ considered so far by a binary operator $\leadsto$, referred to as the *conditional*. As an informal reading of $A \leadsto B$ I suggest the subjunctive *if A were the case, then B would be the case.* (Other authors use the corner $>$ (Stalnaker), the box-arrow $\Box\!\!\rightarrow$ (Lewis), or the square horseshoe $\sqsupset$ (Segerberg) for the conditional. The symbol $\leadsto$ suggests – rightly, as it will turn out – that the conditional relates to cumulative consequence ($\mid\!\sim$) in much the same way as material implication ($\rightarrow$) relates to classical consequence ($\vdash$).)

The key ingredient of the possible worlds semantics adopted here is a propositional selection function (as in [2]). This is meant to reflect the fact that sometimes the truth of certain sentences is to be evaluated at a set of *preferred* ("selected") worlds. I take this to be both the formal and intuitive core of the minimal models approach to nonmonotonic reasoning. Those familiar with the minimal models approach of Shoham and others may want to think of the selection function as selecting those worlds that are minimal under some underlying preference ordering. Note that this is only a rough recommendation: a selection function generated from a partial order on the set of worlds will assume certain properties which we shall not always impose.

The use of a distinguished world (the "real world") unifies – as will emerge below – the semantics of conditionals and of cumulative inference. It also gives semantic substance to the claim that $\mid\!\sim$ stands for some notion of *plausible consequence*. To draw plausible conclusions one needs a basis for distinguishing between the plausible and the implausible. In our models the requisite standard of plausibility is determined by what is true (simpliciter), i.e. by what holds in the real world.

The following system is a natural basis for building up conditional logics. Note the similarity to the formulation of the cumulative logic **CB** in the previous section.

System $\mathbf{CB}^{\mid\sim}$ (basic cumulative logic). $\dfrac{A \dashv\vdash B}{A \rightsquigarrow C \vdash B \rightsquigarrow C}$

$\dfrac{B \vdash C}{A \rightsquigarrow B \vdash A \rightsquigarrow C}$ Right Weakening*

$(A \rightsquigarrow B) \wedge (A \rightsquigarrow C) \vdash A \rightsquigarrow (B \wedge C)$ And*

$A \rightsquigarrow \top$ Truth*

DEFINITION 1. A *frame* is a structure $(W, 0, P, s)$ where W is a nonempty set of points (worlds) with $0 \in W$, and $P \subseteq 2^W$ is a set of propositions such that

(i) $W \in P$,
(ii) if $X \in P$, then $W \backslash X \in P$,
(iii) if $X, Y \in P$, then $X \cap Y \in P$,
(iv) if $X, Y \in P$, then $\{a \in W : s(X, a) \subseteq Y\} \in P$,

$s : P \times W \rightarrow 2^W$ selects a set of worlds in terms of a proposition *cum* world.

The intuition underlying the selection function is that in evaluating a conditional at a world a s selects among the worlds that satisfy the antecedent of the conditional those that differ the least from a.

DEFINITION 2. A *model* on a frame is a structure $(\mathcal{F}, I)$, where $\mathcal{F}$ is a frame, and $I : Fml \times W \to \{0,1\}$ is an assignment of truth-values to formulae at worlds. I shall write $|A|$ for a proposition expressed by a formula A, i.e. $|A| := \{a \in W : I(A,a) = 1\}$. The assignment I is to respect the usual conditions for boolean connectives:

$$|\top| = W, \ |\neg A| = W \backslash |A|, \ |A \wedge B| = |A| \cap |B|$$

For the conditional connective $\rightsquigarrow$ we require

$$I(A \rightsquigarrow B, a) = 1 \text{ iff } s(|A|, a) \subseteq |B|$$

for all $a \in A$ and formulae A and B.

Thus a conditional $A \rightsquigarrow B$ is true in a world a just in case B holds throughout those A-worlds that are most similar to a.

The definitions of truth in a world (on a frame, in a class of frames or models) are the standard ones.

A formula A is **true in a model** $\mathcal{M}$ iff $I(A, 0) = 1$, i.e. iff A is true in the real world of that model. Notation: $\mathcal{M} \models A$.

A is **true in a frame** $\mathcal{F}$ iff A is true in all models $\mathcal{M} = (\mathcal{F}, I)$ on $\mathcal{F}$. Notation: $\mathcal{F} \models A$.

A is true in a class $\mathcal{X}$ of frames or models (A is *valid* in $\mathcal{X}$) iff A is true in each frame or in each model in $\mathcal{X}$. Notation: $\mathcal{X} \models A$.

THEOREM 3. *Let FRM be a class of all frames and let $MOD^{\rightsquigarrow}$ be the class of all models (in the just defined sense) on frames in FRM. Then $MOD^{\rightsquigarrow} \models A$ if and only if A is a theorem of* $\mathbf{CB}^{\rightsquigarrow}$*, for all formulae A.*

In the direction (soundness) the theorem is proved by an easy induction on the length of a derivation in $\mathbf{CB}^{\rightsquigarrow}$. The other direction (completeness) may be proved by constructing a suitable canonical model.

Interlude: Flat models on flat frames

We have first introduced $\mid\!\sim$ as a relation between formulae. As such it is not part of the formal language $\mathcal{L}$ but an expression of (mathematical)

English. But note that, since all sequences are singular on the left (and on the right), nothing in our formulation of cumulative logics requires that $\mid\!\sim$ be treated in this way. We could have used the rules of cumulative inference without ever noticing the difference between $\mid\!\sim$ treated as a relation and $\mid\!\sim$ treated as a connective with, perhaps, somewhat restrictive formation rules.

We shall now adapt the semantics of the last section to model cumulative systems of inference in a language $\mathcal{L}^{\mid\sim}$ in which $\mid\!\sim$ holds syntactically just like a relation but is in fact interpreted as a partial connective. The system $\mathbf{CB}^{\mid\sim}$ is determined by the axioms an rules for **CB** but these postulates are now to be understood as formulated in the language $\mathcal{L}^{\mid\sim}$.

Definition 4. The *primitive* constituents of the language $\mathcal{L}^{\mid\sim}$ are connected in three sets:

$Atm = \{p, q, r, \ldots\}$ is the set of **atomic formulae**,

$BOpr = \{\neg, \wedge\}$ is the set of **boolean operators**,

$COpr = \{\mid\!\sim\}$ consists of the **conditional operator**,

(1) $BFml$ (**boolean formulae**) is the smallest set such that $Atm \subseteq BFml$ and
if $A, B \in BFml$ then $\neg A, (A \wedge B) \in BFml$.

(2) $CFml$ (**conditional formulae**) is the smallest set such that if $A, B \in BFml$ then $A \mid\!\sim B \in CFml$.

(3) The set Fml of **formulae** is the union of $BFml$ and $CFml$.

The effect of this definition is to make $\mid\!\sim$ a connective that behaves syntactically just like $\vdash$: no nesting and no boolean composition of $\mid\!\sim$-formulae.

Definition 5. A **flat frame** is a structure $\mathcal{F} = (W, 0, P, s)$ where

W is a nonempty set (of worlds); variables: $a, b, c, \ldots$;

$0 \in W$ represents the "real world";

$P \subseteq 2^W$ is a set of propositions such that $W \in P$, and if X and Y are propositions, so are $X \cap Y$ and $W \backslash X$;

$s : P \times W \rightarrow 2^W$ is a selection function.

Definition 6. A **flat model** on a flat selection frame is a structure $\mathcal{M} = (\mathcal{F}, I)$ where

$\mathcal{F}$ is a selection frame and

$I : Fml \times W \to \{0,1\}$ is a (flat) assignment satisfying the following conditions. (As before we put $|A| := \{a \in W : I(A,a) = 1\}$.) For any $a \in W$ and $A, B \in BFml$:

$$|\top| = W,\ |\neg A| = W \backslash |A|,\ |A \wedge B| = |A| \cap |B|$$

$$I(A \mid\!\sim B, 0) = 1 \text{ iff } s(|A|, 0) \subseteq |B|$$

In contrast to the kind of frames defined in the previous section ("full" frames) flat frames are less restrictive as to the set P of propositions. Moreover in flat models nothing is known about the truth of conditional formulae in worlds other than the real world.

The notions of truth in a model (on a frame, in a class of frame) are defined as before.

Let $\flat FRM$ be the class of all frames and let $\flat MOD^{\mid\sim}$ be the class of all flat models on flat frames.

THEOREM 7. *For each formula* $A : \flat MOD^{\mid\sim} \models A$ *if only if* A *is a theorem of* $\mathbf{CB}^{\mid\sim}$.

(The proof of the theorem proceeds in much the same way as the one for Theorem 3. Since the set of formulae is not closed under negation, some care has to be taken with the Lindenbaum construction of points for the canonical model.)

Plain models on flat frames

Let us now return to our original simple language $\mathcal{L}$: no conditionals, only truth-functional connectives. But, as before, we want to look at two notions of inference – denoted by $\vdash$ and $\mid\!\sim$ – interacting in one formal system. Such systems, as it will turn out, may be semantically represented by plain models on flat frames.

DEFINITION 8. A **plain model on a flat frame** is a structure $(\mathcal{F}, I)$ where $\mathcal{F}$ is a flat frame (see definition 5) and I is an assignment $Fml \times W \to \{0,1\}$ such that for all formulae A and B,

$$|\top| = W,\ |\neg A| = W\backslash|A|,\ |A\wedge B| = |A|\cap|B|$$

Let $\flat MOD$ be a class of all plain models on flat frames. The definitions of truth in a model (on a frame in a class of frames) are the standard ones given before. We define two notions of consequence relative to plain models on flat frames.

B is a (plain) **consequence** of A in a model $\mathcal{M} \in \flat MOD$ if and only if $|A| \subseteq |B|$ in $\mathcal{M}$. Notation: $A \models B[\mathcal{M}]$.

B is a **plausible consequence** of A in $\mathcal{M} \in \flat MOD$ if and only if $s(|A|, 0) \subseteq |B|$ in $\mathcal{M}$. Notation: $A \models\!\!\!\approx B[\mathcal{M}]$.

These notions are extended to frames and classes of frames in the usual way.

The definitions of the two consequence relations may be rephrased as follows. B is a consequence of A just in case every way of making A true is a way in which B will turn out true. B is a plausible consequence of A just in case every plausible way of making A true is also a way which makes B true. What is plausible in a model depends on what is true in that model – it depends on what holds in the real world of that model.

THEOREM 9.

(1) $A \vdash B$ *in* **CB** *if only if* $A \models B[\mathcal{M}]$ *for every* $\mathcal{M} \in \flat MOD$.

(2) $A \mid\!\sim B$ *in* **CB** *if and only if* $A \models\!\!\!\approx B[\mathcal{M}]$ *for every* $\mathcal{M} \in \flat MOD$.

Just as stronger systems of cumulative inference may by assembled by combining further postulates with the basic logic **CB**, so corresponding classes of models may be defined by combining suitable conditions. Such correspondences are usually easy to find and, in most cases worthy of consideration, well-known from the literature on conditionals.

The relationship between the basic cumulative system **CB**, the basic system with flat conditionals $\mathbf{CB}^{\mid\sim}$, and the basic conditional logic $\mathbf{CB}^{\leadsto}$ is summarized in the next theorem.

THEOREM 10.

(1) $A \mid\!\sim B$ *in* **CB** *if and only if* $A \mid\!\sim B$ *in* $\mathbf{CB}^{\mid\sim}$*;*

(2) if $A \mid\!\sim B$ *in* $\mathbf{CB}^{\mid\sim}$*, then* $A \leadsto B$ *in* $\mathbf{CB}^{\leadsto}$*;*

(3) for boolean formulae A *and* B*: if* $A \leadsto B$ *in* $\mathbf{CB}^{\leadsto}$*, then* $A \mid\!\sim B$ *in* $\mathbf{CB}^{\mid\sim}$*.*

(Proposition (1) and (2) of the theorem follow straightforwardly from syntactic considerations. The argument for (3) is based on two observations. First, the class FRM of frames is a subset of a set $\flat FRM$ of flat frames. Second, the interpretation of conditionals, $A \rightsquigarrow B$ and a flat conditionals, $A \mid\!\sim B$, in models on frames, respectively flat frames, coincide whenever A and B are Boolean.)

Finally we note that **CB**, $\mathbf{CB}^{\mid\sim}$ and $\mathbf{CB}^{\rightsquigarrow}$ are *decidable*. This follows immediately from Segerberg's and Strevens' [4] observation that canonical models for $\mathbf{CB}^{\rightsquigarrow}$ can be filtered down to finite models.[1]

References

[1] Kraus, Lehmann and Magidor, *Nonmonotonic reasoning, preferential models and cumulative logics*, **Artifical Intelligence** vol. 44 (1990), pp. 167–207.

[2] D. Lewis, **Counterfactuals**, Oxford (Blackwell) 1973.

[3] D. Nute, **Topics in Conditional Logic**, Dordrecht (Reidel), 1980.

[4] K. Segerberg, *Notes on conditional logic*, **Studia Logica** vol. 48 (1989), pp. 157–168.

[5] R. Stalnaker, *A theory of conditionals*, in **Studies in Logical Theory**, American Philosophical Quarterly Monograph Series, No. 2, Oxford (Blackwell), 1968.

Department of Philosophy
University of Konstanz
P.O. Box 5560, 7750 Konstanz
Germany

[1] A longer version of this paper is available as a Technical Report: *Konstanzer Berichte* No 14, February 1991. The full paper will also appear in a volume edited by D. Pearce for *Lecture Notes in Artificial Inteligence.*