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A FINITARY RELATIONAL ALGEBRA FOR CLASSICAL FIRST-ORDER LOGIC*

Abstract

A simple extension of Tarski's calculus of binary relations is shown to have the expressive power of first-order logic. In contrast with Cylindric and Polyadic algebras, this extension introduces only finitely many operations and constants whose standard meaning is finitary.

1. Introduction

This paper shows that a simple extension of the well-known calculus of binary relations can have the expressive power of first-order logic. This seems to have been one of Tarski's intentions in developing his *Calculus of Binary Relations* [5], whose expressive power, however, is weaker than that of first-order logic. Subsequent efforts to attain the expressive power of first-order logic [4,3] led to abandoning binary relations and to introduction of infinitely many new operations. Our extended calculus was originally motivated by the needs of program derivation [2]. Programs can be regarded as solutions to problems, which involve binary input-output relations [6]. But, a calculus of programs should manipulate them as objects by means of their relational properties, rather than as sets of input-output pairs.

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This paper starts by revieving Tarski's theories of binary relations and the problem of expressiveness. Then, we introduce our extension to binary relations over strings (finite sequences) and show how they can represent sets of strings. The central result shows that if a set can be defined be a first-order formula, then its representation can be denoted by a relational term. Finally, we analyse our extension, comparing it with the extant algebras of logic.

2. Tarski's theories of binary relations and the problem of expressiveness

Tarski develops his *Elementary Theory of Binary Relations* as an extension of first-order logic by introducing variables ranging over two sorts: *individuals* (denoted here by $x, y, z, \ldots$) and *relations* (denoted here by italic letters $r, s, t, \ldots$). The atomic sentences are of the form $r(x, y)$ (with intended meaning "x is in relation r with y") and $r \approx s$ (where the symbols $\approx$ denotes equality). As usual, the compound sentences are obtained from atomic ones by means of logical connectives $\wedge, \vee, \leftrightarrow, \rightarrow, \neg$, and the quantifiers $\forall$ and $\exists$. In the standard models of the *Elementary Theory of Binary Relations*, variables on individuals range over a fixed set A, while variables on relations range over some subsets of $A \times A$.

The symbols introduced by Tarski are, in our notation, ∞ (for the universal relation), 0 (for the null relation), 1 (for the indentity relation) and $\mathcal{P}$ (for diversity relation), as relational constants, together with the following operations on relations: $^-$ (complement), $^\sim$ (converse), $+$ (union), $\bullet$ (intersection), $\oplus$ (relative addition), and ; (relative product). The symbols $\infty, 0, {}^-, +$ and $\bullet$, are called *absolute* or *Boolean*, whereas $1, \mathcal{P}, {}^\sim, \oplus$ and ;, are called *relative* or *Peircean*.

To develop his *Calculus of Binary Relations* Tarski [5] derives from the axioms of his *Elementary Theory of Binary Relations* an appropriate set of theorems whose variables are exclusively relational ones. He then takes these theorems as extralogical axioms of his *Calculus of Binary Relations*. These extralogical axioms can be divided into three groups: axioms of the absolute or Boolean symbols (axioms for *Boolean Algebras*, e.g. $(r+s) \bullet t \approx (r \bullet t) + (s \bullet t)$), axioms of the relative or Peircean symbols (e.g. $\widetilde{r;s} \approx \widetilde{s};\widetilde{r}$), and axioms relating absolute and relative symbols (e.g. $(r;s) \bullet \widetilde{t} \approx 0 \rightarrow$

$(s;t) \bullet \widetilde{r} \approx 0)$. Shorter axiomatization have been provided, for instance, by Chin, Jónsson and Tarski.

Thus the standard models of Tarski's *Calculus of Binary Relations* are the *proper relational algebras*, i.e. algebras of the form $\mathbf{T} = < \Re, +, \bullet, ^-, \infty, 1, ;, ^\sim >$, where $< \Re, +, \bullet, ^-, \infty, 1 >$ is a *Boolean Algebra* of subsets of $A \times A$, for some set A, closed under the Peircean operations, that satisfy the axioms of his calculus.

In examining the expressiveness of his *Calculus of Binary Relations*, Tarski [5] asks whether every property of relations, relation among relations, etc., that can be defined in his *Elementary Theory of Binary Relations*, can be expressed in his calculus. Tarski's answer to this question is negative. According to him, even a simple expression like:

$$\forall x \forall y \forall z \exists u (r(x,u) \wedge r(y,u) \wedge r(z,u)) \tag{1}$$

cannot be expressed within his relational calculus. For instance, no sentence of the relational calculus is satisfied by exactly the same relations that satisfy expression (1).

Since the *Elementary Theory of Relations* is an extension of first-order logic, the problem of expressiveness posed by Tarski leads to the question: can the *Calculus of Binary Relations* be finitely extended so as to bear the same relationship to first-order logic that *Boolean Algebras* bear to sentential logic? In this direction subsequent works of Chin, Henkin, Monk, Tarski and Thompson [4] led to the introduction of *Cylindric Algebras*. But, in doing this they abandon the algebras of *binary relations* in favour of algebras of *sets of infinitary sequences*, and likewise for Halmos's *Polyadic Algebras* [3].

3. Algebras of binary relations over strings

Tarski's theories of binary relations are theories of subsets of $A \times A$. We will extend them by (i) giving a structure to our *universe*, and (ii) introducing two new operations.

To accomplish extension (i), we take our *universe* $\mathcal{U}$ as being the *free monoid* B^* over a given set B, with operation denoted by $*$ and with neuter element denoted by λ. B will be referred to as the *base set* of $\mathcal{U}$. So, elements of $\mathcal{U}$ will be finite sequences (strings) $[x_1, \ldots, x_n]$ with

$x_1, \ldots, x_n \in B$. As for extension (ii), we introduce two new (structural) operations on relations, namely *fork* (denoted by ∇) and *concatenation* (denoted by $\times$), defined by

$$r \nabla s = \{< x, y * z >: < x, y > \in r \wedge < x, z > \in s\}$$

$$r \times s = \{< x * y, z * t >: < x, z > \in r \wedge < y, t > \in s\}$$

Thus, the standard models of our *Extended Calculus of Binary Relations* will be algebras of the form

$$\mathbf{U} = < \Re, +, \bullet, ^-, \infty, 1, 1_B, 1_\lambda, ;, ^\sim, \nabla, \times >,$$

where $< \Re, +, \bullet, ^-, \infty, 1 >$ is a proper relational algebra over $\mathcal{U} = B^*, 1_B$ is the identity relation on strings over B with length 1, and 1_λ is the identity on $\{\lambda\}$. It is interesting to notice that $< \Re, +, \bullet, ^-, \infty, 1, ;, ^\sim >$ is $\mathbf{T}$ where $\mathcal{U}$ plays the role of the set of "points". In other words, by means of $1_B, 1_\lambda, ;, ^\sim, \nabla$ and $\times$, one is able to "resolve" $\mathcal{U}$ into a hierarchy of strings (of elements of B) of increasing length.

It is instructive to examine how we manage to express (1) within our formalism and how we can overcome the difficulties encountered in trying to express it in Tarski's relational calculus. The key idea is the 'simulation' of the existential quantifier $\exists u$ by the relative product. If we apply this simple idea to (1), we are led to something like $\forall x \forall y \forall z \exists u (r(x,u) \wedge \widetilde{r}(u,y) \wedge \widetilde{r}(u,z))$. This is equivalent to (1), but we cannot simulate the effect of $\exists u$ by the relative product. The reason for this is the fact that variable u now occurs three times in the matrix of the formula. Tarski's relational calculus has no variables over individuals, and the relative product $r; \widetilde{r}$ "consumes", so to speak, variable u. We overcome this difficulty by means of the fork (which also has a character of existential quantification, as can be seen from its definition, a viewpoint that will be used later on.). Thus, we see that the addition of extra operations and the above definition of the universe of discourse increase the expressiveness of the language of relations.

Encouraged by these positive results, one is tempted to conjecture that our *Extended Calculus of Binary Relations* has the expressive power of first-order logic. We will show that indeed this is so.

4. Definable subsets and their representations

Consider a *first-order language* $\mathcal{L}$ with equality $\approx$, given by a set $\mathcal{P}$ of *predicate symbols* (we do not consider functions symbols, since functions can, for our purposes, by replaced by their graphs). As usual, a *structure* C for $\mathcal{L}$ consists of a *domain* C together with a realization $p^{\mathsf{C}} \subseteq C^n$, for each n-ary predicate symbol p in $\mathcal{P}$. This can be extended to assign a realization for each formula $\phi \in \Phi(\mathcal{L})$ (where $\Phi(\mathcal{L})$ is the set of formulae of $\mathcal{L}$). It is more convenient, however, to view a formula as defining a set of strings over C. For this purpose we introduce some notation to be used also later. We let $f(\phi)$ be the set of variables with free occurrences in formula ϕ, and for a finite set Y of variables, we let $h(Y) = max\{i : v_i \in Y\}$. Consider a formula $\phi \in \Phi(\mathcal{L})$ with $h(f(\phi)) = n$; we associate with ϕ a set ϕ^{C} of strings over C defined as follows

$$\phi^{\mathsf{C}} = \{[c_1, \ldots, c_n] : \mathsf{C} \models \phi(c_1, \ldots, c_n)\}$$

where $\mathsf{C} \models \phi(c_1, \ldots, c_n)$ means that the assignment of c_i to v_i, for $i = 1, \ldots, n$, satisfies ϕ in C [1]. We call a subset $\mathcal{S}$ of $\mathcal{U} = C^*$ *definable* iff $\mathcal{S} = \phi^{\mathsf{C}}$ for some formula $\phi \in \Phi(\mathcal{L})$. We shall denote by $\mathcal{E}(\mathsf{C})$ the set of all finite unions of definable subsets of $\mathcal{U}$. Clearly, $\mathcal{E}(\mathsf{C})$ is a *Boolean Algebra* of subsets of $\mathcal{U}$.

A structure C for $\mathcal{L}$ assigns to each n-ary predicate symbol p an n-ary relation p^C. In addition, we wish to regard a structure as assigning a binary relation (on strings) to each predicate symbol. Thus, we define the *relational realization* of predicate p in the structure C as

$$\mathsf{C}[p] = \{< [c_1, \ldots, c_n], \lambda >: < c_1, \ldots, c_n > \in p^{\mathsf{C}}\}.$$

The above relational interpretation of predicate symbols suggests using them to build other relations by means of the relational operations. The set $T(\mathcal{L})$ of *relational terms* is obtained from the *basic terms*, $1, 1_C, \infty$, and p (for each $p \in \mathcal{P}$), by means of the *relational operation symbols* $+, \bullet, ^-, ;, ^\sim, \nabla, \times$. (This is the set of *closed terms*, the set of terms with variables in $\mathcal{W}$ has, in addition, all variables in $\mathcal{W}$ as basic terms.)

Now, given a structure C for $\mathcal{L}$, with domain C, each such term $\dagger$ denotes binary relation $\mathsf{C}[\dagger]$ on $\mathcal{U} = C^*$ defined inductively in the obvious way (e.g. $\mathsf{C}[\widetilde{\dagger}] = \widetilde{\mathsf{C}[\dagger]}$.) We call a binary relation $r \subseteq \mathcal{U} \times \mathcal{U}$ *denotable* iff $r = \mathsf{C}[\dagger]$ for some $\dagger \in T(\mathcal{L})$, and use $\mathcal{D}(\mathsf{C})$ to refer to the set of all denotable relations over C.

In order to correlate the elementary subsets in $\mathcal{E}(\mathsf{C})$ with the denotable relations in $\mathcal{D}(\mathsf{C})$, we need a way of representing a subset $\mathcal{S} \subseteq \mathcal{U}$ by a binary relation $r \subseteq \mathcal{U} \times \mathcal{U}$. The relational interpretation of a predicate symbol suggests representing the set $\mathcal{S} \subseteq \mathcal{U}$ by the relation $\mathcal{S} \times \{\lambda\}$. Let $\Lambda(\mathcal{U})$ be the set of all binary relations r over $\mathcal{U}$ with $Ran(r) = \{\lambda\}$. We define $\Lambda : \mathcal{P}(\mathcal{U}) \to \Lambda(\mathcal{U})$ by $\Lambda(\mathcal{S}) = \mathcal{S} \times \{\lambda\}$. This representation is quite natural, in that $\Lambda(p^{\mathsf{C}}) = \mathsf{C}[p]$; but another one will prove more convenient. Let $I(\mathcal{U})$ be the set of all identity relations over $\mathcal{U}$ and define $I : \mathcal{P}(\mathcal{U}) \to I(\mathcal{U})$ by assigning to $\mathcal{S}$ the identity relation over $\mathcal{S}$.

Some simple properties of these representations will be of interest. First, both representations of sets are equivalent, via a bijection $\mathbf{b} : \Lambda(\mathcal{U}) \to I(\mathcal{U})$ and its inverse. Second, these bijections can be represent by terms, $\mathbf{b}$ being represented by the term $b(w) = (w, \infty) \bullet 1$. Thus, from the viewpoint of denotable relations, they are interchangeable. Finally, both representations are bijective, their inverses assigning to each such relation its domain. In fact, it is interesting to notice that $x \in \mathcal{S}$ iff $< x, x > \in I(\mathcal{S})$. Thus, $\mathcal{S}$ is elementarily definable iff $I(\mathcal{S})$ is so, and both by prenex formulae with the same prefix [1].

5. Main result: expressiveness

We are now ready for our main result, which will show that terms in our *Extended Calculus of Binary Relations* on strings have the expressive power of first-order logic. This will be established by showing that any set of strings that can be defined by a first-order formula can also be denoted be a closed term of our *Extended Calculus Binary Relations* on strings. In view of the remarks in the preceding section, concerning representation of subsets as relations, it suffices to show that, for every set in $\mathcal{E}(\mathsf{C})$, there exists a term † in $T(\mathcal{L})$, such that $\mathsf{C}[\dagger]$ is the identity on $\mathcal{S}$. So, let us denote by $\Im(\mathcal{L})$ the set of all terms † in $T(\mathcal{L})$, such that, for every structure C for $\mathcal{L}, \mathsf{C}[\dagger] \subseteq \mathsf{C}[1]$. Also, we define, by induction on $m, e_1 = 1_C$ and $e_m = e_{m-1} \times 1_C$.

THEOREM *Given a first-order language $\mathcal{L}$, there exists a function $\mathcal{T} : \Phi(\mathcal{L}) \to \Im(\mathcal{L})$, such that for every structure C for $\mathcal{L}, I(\phi^{\mathsf{C}}) = \mathsf{C}[\mathcal{T}(\phi)]$, for every formula $\phi \in \Phi(\mathcal{L})$.*

PROOF. We will define $\mathcal{T}$ by induction on the structure of formula ϕ.

Basis: Let us distinguish three cases.
Case 1: ϕ is $v_1 \approx v_2$. Then, we set $\mathcal{T}(v_1 \approx v_2) = \widetilde{2}; 1_C; 2$, where $2 = 1\nabla 1$.
Case 2: ϕ is $p(v_1, \ldots, v_n)$ with $p \in \mathcal{P}$. Then, we set $\mathcal{T}(p(v_1, \ldots, v_n)) = (p; \widetilde{p}) \bullet 1$.
Case 3: ϕ is an atomic formula $\alpha(u_1, \ldots, u_m)$ obtained from one of the above $\alpha(v_1, \ldots, v_n)$ by means of a substitution σ. Then, Lemma 2 below yields $\mathcal{T}(\alpha(u_1, \ldots, u_m)) = \widetilde{s}; \mathcal{T}(\alpha(v_1, \ldots, v_n)) = s$, where $\widetilde{s}$ is a term, simulating σ, easily constructed by means of $1, 1_C$ and ∇.
Inductive step
If ϕ is $\neg\psi$, then we set $\mathcal{T}(\neg\psi) = \overline{\mathcal{T}(\psi)} \bullet e_n$.
Now, let $h(f(\phi)) = n$. In view of Lemma 1 below, it suffices to consider ϕ' of the form $\phi \wedge v_1 \approx v_1 \wedge \ldots \wedge v_n \approx v_n$.
If ϕ is $\psi \vee \theta$, then ϕ' is equivalent to $\psi' \vee \theta'$, where ψ' is $\psi \wedge v_1 \approx v_1 \wedge \ldots \wedge v_n \approx v_n$ and similarly for θ'. Lemma 1 applied to the inductive hypothesis gives $\mathcal{T}(\psi')$ and $\mathcal{T}(\theta')$. We then set $\mathcal{T}(\psi \vee \theta) = \mathcal{T}(\psi') + \mathcal{T}(\theta')$. If ϕ is $\exists v_i \psi$, then clearly ϕ' is equivalent to $\exists v_n \psi' \wedge v_n \approx v_n$, where ψ' is obtained from ψ by the substitution σ that interchanges v_i and v_n. So, lemma 2 below, applied to the inductive hypothesis, gives $\mathcal{T}(\psi')$. Then, with $\widetilde{\Pi_1} = 1_C \nabla \infty$, we set $\mathcal{T}(\exists v_n \psi') = \widetilde{\Pi_1}; \mathcal{T}(\psi'); \Pi_1$, whence Lemma 1 will give $\mathcal{T}(\phi)$. □

Now, we define, by induction on $n, d_0 = \infty; 1_\lambda$ and $d_n = d_{n-1} \nabla \Pi_n$; where the projections Π_j are defined by induction: $\widetilde{\Pi_1} = 1_C \nabla \infty$ and $\widetilde{\Pi_j} = (\infty; 1_C \nabla \widetilde{\Pi_{j-1}})$.

Lemma 1. *For every $m \geq 0$ and for every formula ϕ with $h(f(\phi)) = n$, if ϕ' is the formula $\phi \wedge v_1 \approx v_1 \wedge \ldots \wedge v_{m+n} \approx v_{m+n}$, one can have $\mathcal{T}(\phi') = \mathcal{T}(\phi) \times e_m$ and $\mathcal{T}(\phi) = \widetilde{d_n}; \mathcal{T}(\phi'); d_n$.* □

Lemma 2. *Given a substitution σ on $\{1, \ldots, n\}$, there exists a term $s(\sigma)$, such that for any formula ϕ with $h(f(\phi)) = n$, if $\sigma(\phi)$ is the formula obtained by applying σ to ϕ, then one can take $\mathcal{T}(\sigma(\phi)) = \widetilde{s(\sigma)}; \mathcal{T}(\phi); s(\sigma)$.*

Proof. Let $h(f(\phi)) = n$. By Lemma 1, we may assume $f(\phi) = \{v_1, \ldots, v_n\}$.
Let $m = h(Ran(\sigma))$ be the highest index among $\sigma(1), \ldots, \sigma(n)$. We may assume σ onto $\{1, \ldots, m\}$. (For otherwise, we can replace σ by τ from $\{v_1, \ldots, v_n, \ldots, v_p\}$ onto $\{v_1, \ldots, v_m\}$ and, by Lemma 1, ϕ by ψ of the form $\phi \wedge v_1 \approx v_1 \wedge \ldots \wedge v_n \approx v_n \wedge \ldots \wedge v_p \approx v_p$, so that $\tau(\psi)$ is equivalent

to $\sigma(\phi) \wedge v_1 \approx v_1 \wedge \ldots \wedge v_m \approx v_m$ which is enough in view of Lemma 1.) So, assuming σ onto $\{v_1, \ldots, v_n\}$, we set

$$\widetilde{s(\sigma)} = e_m; \Pi_{\sigma(1)} \nabla \Pi_{\sigma(2)} \nabla \ldots \nabla \Pi_{\sigma(n)} \Box$$

Now, the set of terms $\Im(\mathcal{L})$ and $\{\dagger \bullet 1 : \dagger \in \mathcal{T}(\mathcal{L})\}$ have the same denotations. So, from the very definitions of the operations and constants of our extended relational calculus one can see that, for each term $\dagger \in \Im(\mathcal{L})$, one has a formula $\phi \in \Phi(\mathcal{L})$, with two free variables, defining it, in the sense that for every structure C for $\mathcal{L}, \mathsf{C}[\dagger] = \phi^{\mathsf{C}}$. Here, C^* is the structure obtained from C by replacing its domain C by the corresponding free monoid C^*, so that now strings are assigned to variables. Thus, we have that the representations of the definable subsets of C^* are exactly the *denotable terms* in $\Im(\mathcal{L})$.

6. An analysis of our extension

We will now compare Tarski's *Calculus of Binary Relations, Cylindric Algebras* [4] and *Polyadic Algebras* [3] with our algebras $\mathbf{U} = < \Re, +, \bullet, ^-, \infty, 1, 1_B, 1_\lambda, ;, ^\sim, \nabla \times >$.

To begin our comparative analysis, we should recall that the standard models of Tarski's *Calculus of Binary Relations* are algebras of the form $\mathbf{T} = < \Re, +, \bullet, ^-, \infty, 1, ;, ^\sim >$, whose reduct $< \Re, +, \bullet, ^-, \infty, 1 >$ is a *Boolean Algebra*, $\Re$ is a subset of the powerset of $A \times A$, and the result of adding ; and $^\sim$ may be called a Peircean extension of this *Boolean Algebra.* As mentioned, Tarski's *Calculus of Binary Relations* is insufficient to attain the expressive power of first-order logic and according to Tarski [5], cannot be finitely extended, in a Peircean-like manner, to attain such expressive power.

Thus, both Tarski et al. with the *Cylindric Algebra*, with Halmos, with the *Polyadic Algebras*, abandon the Peircean-like way of extending $< \Re, +, \bullet, ^\sim, \infty, 1 >$ in favour of more direct attack to the problem. Both approaches maintain a *Boolean Algebra* over an universe of elements whose internal structure, if any, is immaterial. By imposing an infinite arity, they produce theories over sequences of infinite length restricted by a locally-finiteness condition. Tarski et al. chose to represent the existential quantifier by infinitely many cylindrifications (one for each variable), and equations $v \approx w$ by means of a doubly infinite sequence of diagonal elements.

Halmos, on the other hand, represents the existential quantifier by means of a single function but introduces infinitely many transformations on the index set. We should notice that both *Cylindric* and *Polyadic Algebras* are models laden with a syntactical artifact, in that the former, by means of each axis, and the latter, by means of the index set, have names for variables over individuals. Thus, neither *Cylindric* nor *Polyadic Algebras* deserve to be classified as belonging to a calculus of relations. They have a somewhat intermediate status between such calculi and elementary theories of relations.

Our approach is to maintain the Peircean-like way of extending the basic *Boolean Algebra*. So, in addition to ; and $^{\smile}$, we introduce $1_B, 1_\lambda, \nabla$ and $\times$, which are constants and operations on binary relations. The reason why we are able to attain the expressive power of first-order logic is that, instead of extending a *Boolean Algebra* over $B \times B$ we are extending a *Boolean Algebra* over $\mathcal{U} \times \mathcal{U}$, where $\mathcal{U}$ is the free monoid B^* over B. Thus, we introduce a hierarchy of strings at the very level of the universe $\mathcal{U}$ of interpretation. In virtue of this fact, an algebra $\mathbf{U} = < \Re, +, \bullet, ^-, \infty, 1, 1_B, 1_\lambda, ;, ^{\smile}, \nabla, \times >$ can be properly classified as belonging to a *calculus of binary relations*, in Tarski's original sense [5], in that it does not require variables ranging over individuals.

As an important consequence of having structured our universes, neither quantifiers nor substitutions are primitive in our approach. They are instead finitely generate from simpler operations, which is not the case in the cylindric and polyadic approaches. Notice that the argument in the proposition in section 5 hinges on stimulating the effect of an existential quantifier, the remaining cases being dealt with by means of appropriate substitutions. In this sense, the argument is close to the spirit of *Polyadic Algebras*. But this was merely a proof strategy to lay bare the simplicity of the ideas underlying the argument. Actually, we can denote the effect of any first-order quantification and substitution by means of relational term.

Our main result yields several important consequences. For instance, a formula $\neg\phi \in \Phi(\mathcal{L})$ is valid iff $\mathcal{T}(\phi) \approx 0$; and a sentence θ is valid iff $\mathcal{T}(\theta) \approx 1_\lambda$. So, a version of our calculus can be axiomatized by finitely many equations. By translating an axiomatization of first-order logic and the equations required to prove the equalities in the proposition of section 5, one can axiomatize the $\Im(\mathcal{L})$-part of our calculus. In order to axiomatize the full calculus, it is easier to replace the monoid concatenation $*$ by pair formation (i.e., replace the free monoid by a free grupoid), with the obvious adaptation of fork and concatenation [2]. For then, the replacement of relation r by $((\widetilde{1 \nabla r}); (1 \nabla r)) \bullet 1$ enables us to represent each relation within $\Im(\mathcal{L})$.

7. Conclusions

Program derivation needs have suggested us extending Tarski's *Calculus of Binary Relations* by structuring its universe and adding two new operations and two new constants, which are first-order definable. In our extended calculus, substitutions and quantifications turn out to be analyzable into simpler building blocks; thus, our calculus differs essentially from *Cylindric* and *Polyadic Algebras*. The main result is the explicit construction of an interpretation of first-order logic into our calculus. Therefore, our extended calculus has the expressive power of first-order logic.

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