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AN UNDECIDABLE PROBLEM FOR REGULAR EQUATIONS

Abstract

This is a continuation of our note [4]. We deal with two kinds of special equations: normal and regular (see [4]). Our aim is to point an undecidable problem on regular (normal) equations. An extended version is submitted to Algebra Universalis.

Preliminaries

Our nomenclature and notation is basically those of [4]. We consider varieties of universal algebras of a type $\tau : T \to N$, where T is a set and N denotes the set of all positive integers. If $\mathbf{A}$ is an algebra and V is a variety (of type τ), then $E(\mathbf{A})$ and $E(V)$ denotes the set of all equations of type τ, satisfied in $\mathbf{A}$ and V, respectively. Following Płonka, an equation $p = q$ is called *regular* if the set of all variables occurring in p and q coincides. An equation is called *normal*, if p and q are the same variables or neither p nor q is a variable. For a variety $V, R(V)$ and $N(V)$ denotes the set of all regular (normal) equations satisfied in V, respectively. Analogously, for an algebra $\mathbf{A}$ we define sets $R(\mathbf{A})$ and $N(\mathbf{A})$. If $E(V) = R(V)$ $(E(V) = N(V))$, for a variety V, then V is called *regular* (*normal*). From now on we consider only varieties of a finite type τ, i.e. such that T is a finite set.

The aim of this paper is to present a proof of undecidability of a property (R) for regular equations ((N) for normal equations). Following Markov [5] we consider finite associative systems, the elements of which are "words" – i.e. strings of letters belonging to a finite alphabet. Each system is defined by a finite number of generating relations of the form $P \Leftrightarrow Q$, where P and Q are words.

A property $\mathcal{P}$ of associative systems is called invariant if every system which is isomorphic to a system possessing the property $\mathcal{P}$ itself possesses this property. Let $\mathcal{P}$ be an invariant property such that (1) there is a system $\mathbf{S}_0$ which does not have the property $\mathcal{P}$ and is not isomorphic to a subsystem of any system having the property $\mathcal{P}$, and (2) there is a system $\mathbf{S}_1$ which has the property $\mathcal{P}$.

The main result established in [5] (cf. [6]) is that for no property P satisfying conditions (1) and (2) does there exist an algorithm permitting one to decide in a finite number of steps whether an arbitrary given associative system does or does not possess the property $\mathcal{P}$. If $\mathcal{P}$ is an hereditary property, i.e. if every subsystem of a system with the property $\mathcal{P}$ always has the property $\mathcal{P}$, then the condition (1) can be simplified: it is sufficient to assume that there exists a system not having the property $\mathcal{P}$.

Properties satisfying conditions (1) and (2) are referred as Markov's properties.

Following [2] we study some problems of equational theories by transforming them into problems in monoids. A regular equational theory E is called "monadic" iff there exists a presentation of this theory such that all terms in it are built by unary functional symbols only. We apply Lemma 3.11 of [2] which states the connection between E-equality induced by monadic theory and the equality "$=_M$" in the corresponding monoid M_E.

Given an associative system $\mathbf{A}$. Consider a property $(R1)$ defined as follows:

$$(R1) \qquad E(\mathbf{A}) \neq R(\mathbf{A}).$$

It is obvious that $(R1)$ is a hereditary property and that there exist associative systems with and without this property. For example the trivial monoid (on one letter a and the relation $aa = a$) possesses $(R1)$ and a free monoid (on one letter A and the relation $a = a$) does not possess the property $(R1)$. Therefore the property $(R1)$ for associative systems is undecidable.

From now on, suppose that V is finitely axiomatized equational theory (a variety) of a finite, unary type (i.e. $\tau(T) = \{1\}$). It is well known that regular (normal) equations are closed under Birkhoff's rules of inferences (i) – (iv) (see [1], [3], p. 170). Therefore it is decidable if a variety V is regular (i.e. $E(V) = R(V)$). Namely V is regular iff all equations of an axiomatic Σ of V are regular. This explains, the definition of the property (R) below:

$$(R) \qquad E(V) = R(W) \neq E(W),$$

for a variety W of type τ.

Definition. An equational theory V is called (R) theory iff V possesses the property (R).

Theorem. *The class problem for regular* (R) *theories is undecidable.*

Proof. We show that (R) is Markov's property. Let S be the theory defined by all regular equations of a given finite unary type τ. By Theorem 4 of [7], S is finitely axiomatized monadic regular theory with the property (R), namely $S = R(T_\tau)$, where T_τ denotes the trivial equational theory of the type τ. From the other hand, let V be an equational theory of the type τ, defined by the equation $x = x$. Let S be the monoid of terms associated with S (see [2], p. 13, 14). From an observation of A. Tarski [8], that the lattice of all equational theories of a given (non-empty and non-nullary) type, has no minimal non-zero elements, it follows that S is not an (R) theory (we can also use Theorem 5 of [4], to prove that S does not have the property (R)). Let $\mathbf{S}$ be the monoid associated with S and $\mathbf{A}$ be a monoid. Then if $\mathbf{S}$ is a submonoid of $\mathbf{A}$, then $E(\mathbf{S}) \supseteq E(\mathbf{A})$, so if $E(\mathbf{A}) = R(\mathbf{B}) \neq E(\mathbf{B})$, for a monoid $\mathbf{B}$ then $E(\mathbf{S}) \supseteq R(\mathbf{B}) \supseteq E(\mathbf{S})$ and thus $E(\mathbf{S}) = R(\mathbf{B})$ which is impossible by Theorem 5 of [4]. Therefore $\mathbf{S}$ is not a submonoid of any monoid $\mathbf{A}$ with the property (R). We conclude that the property (R) is undecidable. $\square$

Analogously one can define and consider an undecidable property (N) defined in connection with the notion of normal equation.

References

[1] G. Birkhoff, *On the structure of abstract algebras*, **Proc. Cambride Phil. Soc.**, vol. 31 (1935), pp. 433–454.

[2] H. J. Bürckert, A. Herold and M. Schmidt-Schauss, *On Equational Theories, Unification and Decidability*, in C. Kirchner (ed.) special issue on Unification theory **Journal of Symbolic Computation**, vol. 8, no. 1, 2 (1989), pp. 3–49.

[3] G. Grätzer, **Universal Algebra**, Springer-Verlag (1979).

[4] E. Graczyńska, *Regular equations and unifications theory*, **Bulletin of the Section of Logic** vol. 18 no. 1 (1989), pp. 33–39.

[5] A. Markov, *Impossiblity of algorithms for recognizing some properties of associative systems*, **Dokl. Akad. Nauk SSSR** vol. 77 (1951), pp. 953–956.

[6] A. Mostowski, *Review of [5]*, **Journal of Symbolic Logic** vol. 17 (1952), pp. 151–152.

[7] J. Płonka, *On the sum of a system of disjoint unary algebras corresponding to a given type*, **Bull. Polon. Acad. Sci. Ser. Math.**, vol. 30 (1982) No. 7-8, pp. 305–309.

[8] A. Tarski, *Equational logic and equational theories of algebras*, in H. A. Schmidt, K. Schutte and H. J. Thiele, **Contributions to Mathematical Logic**, North-Holland Publ. Co., Amsterdam (1968).

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