Bulletin of the Section of Logic
Volume 20/2 (1991), pp. 67–70
reedition 2005 [original edition, pp. 67–70]

Marcin Mostowski

DIVISIBILITY QUANTIFIERS

Generalized quantifiers as defined by Lindström can be considered as third order concepts preserved by isomorphisms, that is functions giving for every structure M a relation between relations over M – for isomorphic structures isomorphic relations (in the third order sense). On the other hand every such a concept can be considered as a quantifier. Therefore investigating formalizations of logics with additional quantifiers can be considered as a study of formalizations of those third order concepts.

One of crucial concepts in mathematics is the concept of divisibility. It is also one of simplest known nonelementary concepts. We consider a logic of divisibility $L(D_\omega)$ with additional quantifiers D_n (for $n \geq 2$), where $D_n x\phi(x)$ is interpreted as:

The cardinal number of x such that $\phi(x)$ is divisible by n.

We can express it translating $D_n x\phi(x)$ into second order Σ^1_1-formula:

$$\exists P_1 \exists P_2 \ldots \exists P_n \exists R(\forall y \exists z R(y,z) \& \forall y \exists z R(z,y) \& \forall y \forall z \forall z'(R(y,z) \& R(y,z') \Rightarrow z = z') \& \forall y \forall y' \forall z(R(y,z) \& R(y',z) \Rightarrow y = y') \& \alpha_1 \& \ldots \& \alpha_{n-1} \& \beta_{1,2} \& \beta_{1,3} \& \ldots \& \beta_{n-1,n} \& \forall x(\phi(x) \equiv (P_1(x) \vee \ldots \vee P_n(x)))),$$

where α_i is a formula:

$$\forall y \exists z(P_i(y) \to P_{i+1}(z) \& R(y,z)) \& \forall y \forall z(P_{i+1}(z) \& R(y,z) \to P_i(y)),$$

and β_{ij} is a formula:

$$\forall y \neg (P_i(y) \& P_j(z)).$$

By elimination of quantifiers we can justify the following:

THEOREM 1. *The theory of infinitely many unary relations in $L(D_\omega)$ is decidable.*

THEOREM 2. *The theory of the model $(\omega, 0, s, +)$ in $L(D_\omega)$ is decidable.*

On the other hand because a formula "there are infinitely many x such that $\phi(x)$" can be translated preserving equivalence into

$$D_n x\phi(x)\&\exists y(\phi(y)\&D_n x(x \neq y\&\phi(x)))$$

then by defining standadness of models for theories like PA we have:

THEOREM 3. *The theory of one binary relation in $L(D_\omega)$ is not recursively enumerable.*

The theorems 1, 2 can be proved by elimination of quantifiers. The first theorem can be proved by reduction of all formulae (preserving equivalence) to boolean combination of basic formulae of the form: "X is power of t", and "n divides power of $X_{\epsilon_1} + \ldots X_{\epsilon_s} + t$", where X_{ϵ_i} are so called components, that is if $A_1, \ldots, A_k$ are all unary predicates occurring in a formula ϕ; for $\epsilon : \{1, \ldots, k\} \to \{0, 1\}$ we define a formula X_ϵ, being conjunction with k conjuncts such that i-th conjunct is $A_i(x)$ if $\epsilon(i) = 0$, or $\neg A_i(x)$ if $\epsilon(i) = 1$.

The second theorem can be obtained by generalization of known method (see Presburger 1929) – nontrivial part in this case is an elimination of divisibility quantifiers. However basic sentences of this elimination are the same as in a case of Presburger proof. In both cases we need the following lemma which holds for every unary generalized quantifier Q (particularly for $D_n, \forall, \exists$).

LEMMA 1. *Let Q be an unary generalized quantifier preserving equivalence, that is*

$$\models \forall x(\phi \equiv \psi) \to (Qx\phi \equiv Qx\psi),$$

then

1. *Let ϕ be a formula without free occurrences of x then*

$$\models (\phi\&Qx\psi) \equiv (\phi\&Qx(\phi\&\psi)).$$

2. *Let ϕ be a quantifier free formula then there are formulae $\zeta_1, \ldots, \zeta_k$, $\xi_1, \ldots, \xi_k$ such that for $i = 1, \ldots, k$ ζ_i is a boolean combination of atomic*

formulae with no occurrences of x, and ξ_i is boolean combination of atomic formulae having at least one occurrence of x, and

$$\models Qx\phi \equiv (\zeta_1 \& Qx\xi_1) \vee \ldots \vee (\zeta_k \& Qx\xi_k).$$

Comparing these characterizations of $L(D_\omega)$ with those of $L(H_n)$, $L(H_n)$, $L(F_\omega)$ (pure logics with Henkin or function quantifiers) given in Krynicki-Mostowski 1991 we state the following:

THEOREM 4. *The theory of infinitely many unary relations in $L(H_4)$ is not decidable.*

We cannot state anything similar for $L(F_\omega)$, it seems rather that this theory is equivalent in $L(D_\omega)$ and in $L(F_\omega)$. In Krynicki-Mostowski 1991 it was stated that the theory of identity in both logics $L(D_\omega)$ and $L(F_\omega)$ is equivalent and decidable.

It is known also that full first order arithmetic with addition and multiplication can be finitely axiomatized in terms of successor function only in a logic $L(F_2)$ (see Krynicki-Lachlan 1979). Therefore the logic of divisibility appear to be essentially weaker than those of function or Henkin quantifiers.

Third order concepts even being too strong to be formalized within full second order framework can be represented by their weak versions (see Mostowski 1990). According to weak interpretation $PA(D_\omega)$ is equivalent to PA, then weak and strong semantics are essentially different (by the argument justifying theorem 3, the standard model of PA can be axiomatized by the statement saying that every formula $x < y$ is satisfied only by finitely many x). However in arithmetic of addition all divisibility quantifiers are eliminable. Therefore in the last case weak and strong semantics for divisibility quantifiers are equivalent.

References

[1] M. Krynicki, and M. Mostowski, (1991) *Decidability problems in language with Henkin quantifiers*, to appear in **Annals of Pure and Applied Logic**.

[2] M. Mostowski, (1990), *Arithmetic with the Henkin quantifier and its generalizations*, to appear in **Proceedings of the Premierés journées sur les arithmétitiques faibles**, École Normale Supérieure de Lyon 1990.

[3] M. Presburger, (1929), *Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt*, **Comptes Rendus du I^{er} Congrès des Mathématiciens des Pays Slaves**, Warszawa, pp. 92–101.

Section of Logic
Institute of Philosophy and Sociology
Polish Academy of Sciences
8-marca 8, 90-365 Łódź
Poland