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DECISION PROBLEM FOR LINEAR ORDERINGS IN STATIONARY LOGICS

Let $T_{LO}(aa)$ be a theory of linear orderings in the stationary logic $L(aa)$. It is well-known that $T_{LO}(aa)$ is undecidable [Se 82]. On the other hand the decision problem for the theory $T_{DLO}(aa)$ of the class of finitely determinate linear orderings in $L(aa)$ remains open. The following results contribute to a solution of this problem. A linear ordering $(A, <)$ is said to be ω_1-dense iff for all $a, b \in A, a < b$:

$$|\{x : a < x < b\}| \geq \omega_1$$

$(A, <)$ is called ω_1-discrete iff it does not contain a subordering which is ω_1-dense. Let $L(\omega_1)$ be the smallest class of linear orderings containing 1 and closed with respect to $\alpha + \beta$, ω- and ω^*- sums, ω_1- and ω_1^*-sums and η-sums (η is the order type of the rational numbers).

Proposition 1. [He 91]

(1) Let $\mathcal{A}$ be a linear ordering of power ω_1. Then either $\mathcal{A}$ is ω_1-discrete or there is an ω_1-dense linear ordering $\mathcal{C}$ and ω_1-discrete orderings $\mathcal{A}_c$ $c \in C$, such that $\mathcal{A} = \Sigma_{c\in C}\mathcal{A}_c$

(2) A linear ordering $\mathcal{A}$ of power $\leq \omega_1$ is ω_1-discrete if and only if $\mathcal{A} \in \mathcal{L}(\omega_1)$.

Definition 1.

(1) A linear ordering $\mathcal{B}$ is weakly separable if it is short (i.e. neither ω_1 nor ω_1^* is embeddable in it) and every densely ordered subordering $X \subseteq \mathcal{B}, |X| > 1$, contains a non-empty open interval which is separable.

(2) Let M_{WS} be the smallest class of linear oredrings containing 1 and closed with respect to $\alpha+\beta, \omega$- and ω^*-sums, η-sums and ordered sums over ω_1-dense separable orderings.

PROPOSITION 2. [He 91]

(1) A linear ordering $\mathcal{A}$ is weakly separable if and only if $\mathcal{A} \in M_{SW}$.

(2) Every weakly separable linear ordering is finitely determinate.

(3) The theory $Th_{aa}(LO_{WS})$ in $L(aa)$ of the class LO_{WS} of all weakly separable linear orderings is decidable.

Let M_D be the smallest class of orderings containing 1 and closed with respect to finite sums, $\alpha\cdot\omega, \alpha\cdot\omega^*, \alpha\cdot\omega_1, \alpha\cdot\omega_1^*$ and to the shuffling operation $sh_\eta(\Delta), \Delta$ – a finite set of order types (see [La 66]).

PROPOSITION 3. [He 91] *Let K be the class of all finitely determinate ω_1-discrete linear orderings.*

(1) $Th(M_D) = Th(K)$.

(2) $Th(K)$ is decidable.

An ω_1-dense linear ordering $(A, <)$ is said to be a Specker ordering if it is short and there is no uncountable subset $X \subseteq A$ which is embeddable in the ordering of the reals. $(A, <)$ is a special Specker ordering if it is finitely determinate and satisfies the following sentence from $L(aa) : aaX(lim_s(X) = X)$ ($lim_s(X)$ is the set of both sided limit points of X).

PROPOSITION 4. [He 91]

(1) There is a sentence $\Phi \in L(aa)$ such that for every short and finitely determinate linear ordering $\mathcal{A}$ holds: $\mathcal{A}$ is a Specker ordering iff $\mathcal{A} \models \Phi$.

(2) The $L(aa)$-theory of the class of special Specker orderings is decidable.

Let $sh_\eta(\Delta)$ be the shuffling operation over the type η and $sh_\lambda(\Delta_1, \Delta_2)$, $sh_\delta(\Delta)$ the analogous operations w.r.t. the real ordering λ and to a fixed special Specker ordering δ. Let M be the smallest class of linear orderings containing 1 and closed w.r.t. $\alpha + \beta, \alpha \cdot \omega, \alpha \cdot \omega^*, \alpha \cdot \omega_1, \alpha \cdot \omega_1^*, sh_\eta(\Delta)$, $sh_\delta(\Delta), sh_\lambda(\Delta_1, \Delta_2)(\Delta, \Delta_1, \Delta_2$ – finite sets of order types).

PROPOSITION 5. [He 91]

(1) Every ordering in M^* is finitely determinate.

(2) The $L(aa)$-theory of M^* is decidable.

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