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SMALL NONISOMORPHIC MODELS CAN BE EQUIVALENT IN LONG GAMES

Consider the following type of formulas:

$$(\forall x_\alpha \bigwedge_{i_\alpha \in I} \exists y_\alpha \bigvee_{j_\alpha \in J})_{\alpha<\gamma} \bigwedge \Phi^{\overline{i},\overline{j}}(\overline{x},\overline{y}),$$

where γ is an ordinal greater than 0, the index sets I and J are arbitrary, and for all $\overline{i} \in I^\gamma$ and $\overline{j} \in J^\gamma$, $\Phi^{\overline{ij}}(\overline{x},\overline{y})$ is a set of atomic or negated atomic formulas with variables in $\{x_\alpha|\alpha < \gamma\} \cup \{y_\alpha|\alpha < \gamma\}$. The truth of such formula ϕ in a model A is determined by the semantic game, where on round α player $\forall$ first chooses some $x_\alpha \in A$ and $i_\alpha \in I$, and $\exists$ respond with $y_\alpha \in A$ and $j_\alpha \in J$. After γ rounds $\exists$ wins to chosen elements x_α and y_α satisfy all formulas in $\Phi^{\overline{ij}}$, where $\overline{i}$ and $\overline{j}$ are the sequences chosen by $\forall$ and $\exists$, naturally. Then $A \models \phi$ iff $\exists$ has a winning strategy in the semantic game. (In fact, ϕ is a sentence.) Let $V_{\infty\gamma}$ be the language formed by such sentences.

The Ehrenfeucht-Fraïsé-game of length γ between two models A and B is defined as follows: on round α player $\forall$ chooses x_α from either model and $\exists$ replies with an element y_α from the other model. Let $\{a_\alpha, b_\alpha\} = \{x_\alpha, y_\alpha\}$ be such that $a_\alpha \in A$ and $b_\alpha \in B$. Then $\exists$ wins the game if either γ rounds have been played, the set $\{(a_\alpha, b_\alpha)\}$ is a one-to-one function preserving all relations in the models.

It is easy to prove following known facts:

(i) For any limit ordinal $\gamma > 0$, two models A and B are $V_{\infty k}$-equivalent iff $\exists$ has a winning strategy in the Ehrenfeucht-Fraïsé-game of length γ between A and B.

(ii) Two models A and B of cardinality k are $V_{\infty k}$-equivalent iff they are isomorphic.

Moreover, there are many known constructions of model pairs of cardinality 2^k which are nonisomorphic but $V_{\infty k}$-equivalent for all $\gamma < k^+$. See [Tu] or [Hu], for instance. The question arises whether there are smaller such pairs of models in $2^k > k^+$.

If $k = \omega$, the question is open. In fact, we do not know whether it is consistent with $ZFC + \neg GH$ that there are two nonisomorphic models A and B of cardinality $\aleph_1$ such that $A \equiv B(V_{\infty\omega^3})$.

If the consistency of a Mahlo cardinal is assumed, the case $k = \omega_1$ can be solved.

THEOREM 0.1. *Assume the consistency of a Mahlo cardinal. Then there is a model of* ZFC *in which* $2^{\aleph_1} = \aleph_3$*, and there are two linear orderings* A *and* B *of cardinality* $\aleph_2$ *such that* $A \not\cong B$ *but* $A \equiv B(V_{\infty\gamma})$ *for all* $\gamma < \omega_2$*.*

The proof uses a forcing construction from [Mi] and a linear ordering from [NS].

References

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