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HILBERT ϵ-SYMBOL IN THE PRESENCE OF GENERALIZED QUANTIFIERS

So called Hilbert ϵ-symbol transforms a formula $\phi(x)$ in a term $\epsilon\phi(x)$ with the intended meaning: "some x such that $\phi(x)$, if such x exists, arbitrary (or undefined in some renderings) otherwise", see [HB]. It is extensionally determined in the sense that it satisfies the schema:

$$\forall x(\phi \leftrightarrow \psi) \rightarrow (\epsilon x\phi \leftrightarrow \epsilon x\psi) \tag{1}$$

Its natural semantics is given by second order structures $(\mathbf{A}, F)$ with $\mathbf{A}$ a first order structure, $F : P(A) \rightarrow A$ being a choice function in the non empty subsets of A, and the interpretation $[\epsilon x\phi(x)]^{\mathbf{A}} = F(\phi(x)^{\mathbf{A}})$, see [CHH].

Although it has played an important role in proof theory (Hilbert's original purpose) and it is utilized for example in Bourbaki's axiomatic set theory [B] under the vest of τ, the ϵ-symbol has not been very much considered in model theory; perhaps because its additional expressive power with respect to first order structures seems null, and it is in fact null in a precise sense we explain below. This changes radically if we allow the ϵ-symbol in languages with generalized quantifiers. Take for example the quantifier:

$Q_1E_{xy}\phi(x,y) \equiv$ "ϕ defines an equivalence relation having uncountably many equivalence classes".

It is well known that Q_1E is not definable in $L(Q_1)$; it is not even definable in $L_{\infty\omega}(\mathbf{Thin})$, where **Thin** is the class of quantifiers containing all monadic and ordering quantifiers, among others. However:

$$Q_1E_{xy}\phi(x,y) \equiv eq(\phi) \wedge Q_1x(x = \epsilon y\phi(x,y)),$$

where $eq(\phi)$ says that ϕ is an equivalence relation. This follows easily from (1) above.

Given a regular Lindström logic L, let L^ϵ be the language of L enriched with ϵ, and let ϵL be the extension of L obtained by adding all new Lindström sentences definable by formulae of L^ϵ. These must be the ϵ-*invariant* sentences of L^ϵ, those satisfying for any $\mathbf{A}, F, G : (\mathbf{A}, F) \models \phi \Leftrightarrow (\mathbf{A}, G) \models \phi$. It may be shown that ϵL is a regular extension of L, preserving compactness, axiomatizability and Löwenheim and Hanf numbers. A measure of the strength of ϵL is given by the next result where qL is the congruence closure of L, and ΔL is the closure under Δ-interpolation.

THEOREM $qL \leq \epsilon L \leq \Delta L$.

Hence $\epsilon L_{\omega\omega} \equiv L_{\omega\omega}$ and the same is true for any logic satisfying Δ-interpolation. However, after [C]:

COROLLARY. *If $L \leq L(\mathbf{Thin})$ is a proper extension of $L_{\omega\omega}$, then ϵL is a proper extension of L.*

So, for example: $L(Q_0) < \epsilon L(Q_0) \leq L_{\omega_1 CK_\omega}, L(Q_\omega^{cof}) < \epsilon L(Q_\omega^{cof}) \leq L(aa)$, etc. To finish with the obvious questions: is $qL = \epsilon L$? is $\epsilon L = \Delta L$?

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