Bulletin of the **S**ection of **L**ogic
Volume 20:3/4 (1991), pp. 119–123
reedition 2005 [original edition, pp. 119–123]

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NON-AXIOMATIZABILITY OF GRICE'S IMPLICATURE

The aim of this paper is to test Grice's theory of conversational implication [1], so-called *implicature*, by putting it into operation in the simplest possible formal language, that is, by constructing an adequate zero-order (sentential) logic.

According to Grice, in a serious and fair conversation, the participants are supposed to cooperate with each other in the best way they know. This general rule called the *cooperation principle*, splits into four more concrete rules, called *maxims*, which we quote below in a formulation fit for our purposes:

(QLT) *maxim of quality:*	do not utter a sentence which you do not believe to be true;
(QNT) *maxim of quantity:*	convey maximum expected information know to you;
(REL) *maxim of relevance*	do not use extra-logical terms which are not necessary;
(MAN) *maxim of manner*	let the logical form of your utterance be as simple as possible.

Some of the maxims, literally taken, may come in collision; for example, one cannot convey the full information without being irrelevant. The maxims work as a whole, and not separately. The interrelations between the four maxims are stressed in the following formulation:

If you utter a sentence α then (QLT) you are obliged to believe that α is true, and not (QNT) you are expected not to believe in any sentence β more informative than α unless (REL) that β would either have to involve new, that is, not appearing in α, lexical items (propositional variables, in *case of sentential languages), or,* (MAN) *it would have to be more than* α *complicated in its logical form, that is, simply longer than* α*.*

As it is seen, at least two non-classical sentential connectives are needed while formalizing implicature: *believing* hereafter **B**, and *uttering*, hereafter **U**. So, $\mathbf{B}\alpha$ will be read "the speaker believes that α"; $\mathbf{U}\alpha$ will be read "the speaker has uttered α". We shall then consider two formal languages:

$$\mathcal{S} := < S, \neg, \vee, \wedge, \Rightarrow, \equiv, \mathbf{B} >,$$

$$\mathcal{L} := < L, \neg, \vee, \wedge, \Rightarrow, \equiv, \mathbf{B}, \mathbf{U} >,$$

both with $Var = \{p_1, p_2, \ldots\}$ as the set of sentential variables. As the logic for the *believe* connective we choose the system **LB** of [2]. It is based on two rules: MP (*Modus ponens* – from α and $\alpha \Rightarrow \beta$ to infer β), and RB (from α to infer $\mathbf{B}\alpha$), and has the following axiom schemes ($\alpha, \beta \in S$):

Ax.1 *all instances of tautologies*

Ax.2. $\mathbf{B}\alpha \equiv \mathbf{B}\mathbf{B}\alpha$

Ax.3. $\neg\mathbf{B}\alpha \equiv \mathbf{B}\neg\mathbf{B}\alpha$

Ax.4. $\mathbf{B}\neg\alpha \Rightarrow \neg\mathbf{B}\alpha$

Ax.5. $\mathbf{B}(\alpha \Rightarrow \beta) \Rightarrow (\mathbf{B}\alpha \Rightarrow \mathbf{B}\beta)$.

For α, β in $S, l(\alpha)$ denotes the *length* of α; $Var(\alpha)$ denotes the set of variables in α; $\alpha \models\!\!\rightarrow \beta$ means that $\alpha \Rightarrow \beta \in \mathbf{LB}$ but $\beta \Rightarrow \alpha \notin \mathbf{LB}$. As to **U**, let us take the following axiom, the only one that does not seem controversial ($\alpha, \beta \in L$):

Ax.6. $\mathbf{U}(\alpha \wedge \beta) \Rightarrow (\mathbf{U}\alpha \wedge \mathbf{U}\beta)$.

Now, according to the above interpretation, Grice's maxims are to be put the following way:

Ax.7. $\mathbf{U}\alpha \Rightarrow \mathbf{B}\alpha$

Ax.8. $\mathbf{U}\alpha \Rightarrow \neg\mathbf{B}\beta$ for any $\alpha, \beta \in S$ such that $l(\beta) \leq l(\alpha)$ & $Var(\beta) \subseteq Var(\alpha)$ & $\beta \models\!\!\rightarrow \alpha$.

The *Logic of Implicature*, **LI**, is the system resulting from *all instances in* $\mathcal{L}$ of the above axioms Ax.1 – Ax.8 by applying MP and RB. The following formulas (T1) – (T8) are typical theses of **LI**:

(T1) $\mathbf{U}(\alpha \wedge \beta) \Rightarrow \mathbf{B}\alpha \wedge \mathbf{B}\beta$

(T2) $\mathbf{U}(\alpha \vee \beta) \Rightarrow \neg\mathbf{B}\alpha \wedge \neg\mathbf{B}\beta$

(T3) $\mathbf{U}(\alpha \vee \beta) \Rightarrow \neg\mathbf{B}\neg\alpha \wedge \neg\mathbf{B}\neg\beta$

(T4) $\mathbf{U}(\alpha \vee \beta) \Rightarrow \neg\mathbf{B}(\alpha \wedge \beta)$

(T5) $\mathbf{U}(\alpha \equiv \beta) \Rightarrow \neg\mathbf{B}(\alpha \wedge \beta)$

(T6) $\mathbf{U}(\alpha \Rightarrow \beta) \Rightarrow \neg\mathbf{B}\alpha \wedge \neg\mathbf{B}\beta$

(T7) $\mathbf{U}(\alpha \Rightarrow \beta) \Rightarrow \neg\mathbf{B}\neg\alpha \wedge \neg\mathbf{B}\neg\beta$

(T8) $\mathbf{U}(\alpha \Rightarrow \beta) \Rightarrow \neg\mathbf{B}(\alpha \equiv \beta)$.

In what follows, we shall apply the following notation:

AX – the set of all instances (in $\mathcal{L}$) of Ax.1 – Ax.7 (without Ax.8!);

$Sb(X)$ – the set of all substitutions (in $\mathcal{L}$) of formulas of X;

$Cn_{MP,RB}(X)$ – the set of all consequences of $X \subseteq L$ on the basis of MP and RB;

$\delta_k := p_1 \Rightarrow (p_2 \Rightarrow (\ldots \Rightarrow (p_{k-1} \Rightarrow p_k)\ldots)),\ k = 1, 2, \ldots;$

$\Delta_k := \mathbf{U}\delta_k \Rightarrow \neg\mathbf{B}p_k;$

$Th_k := \{\mathbf{U}\alpha \Rightarrow \neg\mathbf{B}\beta : \alpha, \beta \in S\ \&\ l(\beta) \leq l(\alpha)\ \&\ Var(\beta) \subseteq Var(\alpha)\ \&\ \beta \models\!\!\!\to \alpha\ \&\ Var(\alpha) \subseteq \{p_1, p_2, \ldots, p_k\}\};$

$TH_k := Sb(Th_k)$.

Lemma 1. $\Delta_{k+1} \in (Th_{k+1} - TH_k)$.

Proof. It is obvious that $\Delta_{k+1} \in Th_{k+1}$. Suppose that there are a formula $\mathbf{U}\alpha \Rightarrow \neg\mathbf{B}\beta \in Th_k$ and a substitution $e : \mathcal{L} \stackrel{hom}{\longrightarrow} \mathcal{L}$ such that $\Delta_{k+1} = e(\mathbf{U}\alpha \Rightarrow \neg\mathbf{B}\beta)$. Then β is a variable, say p_i, with $i \leq k$. If α were a variable, too it would have to be p_i (since $Var(\beta) \subseteq Var(\alpha) = \{p_i\}$), which is impossible on the ground of the supposition that $\beta \models\!\!\!\to \alpha$. Hence, $\alpha = p_{j_1} \Rightarrow (p_{j_2} \Rightarrow \ldots (p_{j_n} \Rightarrow p_i)\ldots)$ for some $p_{j_m} \in \{p_1, \ldots, p_k\}$, $m = 1, 2, \ldots, n$. All variables in α have to be pairwise different, and consequently $n < k$. It is immediately seen that no substitution of such an α can be of the form δ_{k+1}. □

Lemma 2. $\Delta_{k+1} \notin Cn_{MP,RB}(AX \cup TH_k)$.

Proof. Let's define a function $h : L \longrightarrow \{0, 1\}$ as follows:

(1) $h(p) = 1$, all $p \in Var$;

(2) $h(\neg\alpha), h(\alpha \vee \beta), h(\alpha \wedge \beta), h(\alpha \Rightarrow \beta), h(\alpha \equiv \beta)$ are defined as usual, that is, for example,

$$h(\alpha \Rightarrow \beta) = \begin{cases} 1 & \text{if } h(\alpha) = 0 \text{ or } h(\beta) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

(3)
$$h(\mathbf{U}\alpha) = \begin{cases} 1 & \text{if } \alpha = \delta_{k+1}, \\ 0 & \text{otherwise;} \end{cases}$$

(4)
$$h(\mathbf{B}\alpha) = \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{if } \alpha = 0. \end{cases}$$

Clearly, $h(\Delta_{k+1}) = 0$. We shall prove that for any formula $\alpha \in Cn_{MP,RB}(AX \cup TH_k), h(\alpha) = 1$. All formulas of the form Ax.1 – Ax.6 take value 1 under h – the easy proof will be omitted.

1° Suppose $h(\mathbf{U}\alpha) = 1$ for some α; then $\alpha = \delta_{k+1}$ and $h(\alpha) = 1$. Hence $h(\mathbf{B}\alpha) = 1$, that is, any formula of the form Ax.7 takes value 1 under h;

2° Let $\mathbf{U}\alpha \rightarrow \neg\mathbf{B}\beta$ be any formula in TH_k, and suppose that $h(\mathbf{U}\alpha) = 1$; then $\alpha = \delta_{k+1}$, which is impossible on the ground of Lemma 1. Hence all formulas in TH_k take value 1 under h;

3° Clearly, if $h(\alpha) = 1$ and $h(\alpha \Rightarrow \beta) = 1$ then $h(\beta) = 1$ and $h(\mathbf{B}\alpha) = 1$, that is, MP and RB both preserve value 1 under h, which concludes the proof. □

Naturally, for any k, $Cn_{MP,RB}(AX \cup TH_k) \subseteq Cn_{MP,RB}(AX \cup TH_{k+1})$. What has actually been proved in Lemma 2 is that, for any k, $Cn_{MP,RB}(AX \cup TH_k) \neq Cn_{MP,RB}(AX \cup TH_{k+1})$. On the other hand, however, the construction of TH_k is such that

$$\mathbf{LI} = \bigcup\{Cn_{MP,RB}(AX \cup TH_i) : i = 1, 2, \ldots\}.$$

We have just found a strictly increasing chain of theories closed under

Sb, the join of which is exactly **LI**. So, if our interpretation of Grice's rules in the zero-order language is correct, according to the famous Tarski's criterion the following holds true:

THEOREM. *Grice's implicature is not finitely axiomatizable in a standard formalization (i.e. with MP and RB as the only rules of inference).*

References

[1] H. P. Grice, *Logic and conversation*, [in:] P. Cole, J. L. Morgan (eds.) **Syntax and semantics 3: Speech acts**, Academic Press, New York 1975, pp. 41–58.

[2] M. Tokarz, *On the logic of conscious belief*, **Studia Logica**, vol. 49 (1990), pp. 321–332.

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