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CHARACTERIZATION OF THE REDUCED MATRICES FOR THE $\{\wedge, \vee\}$-FRAGMENT OF CLASSICAL LOGIC

In this note we will deal with the $\{\wedge, \vee\}$-fragment of classical logic, i.e. the deductive system $\mathcal{L} = \langle \underline{\mathbf{F}}, \vdash \rangle$ where $\underline{\mathbf{F}} = \langle Form, \wedge, \vee \rangle$ is the absolutely free algebra of type (2,2) and $\vdash$ is semantically determined by the matrix $\langle \underline{\mathbf{2}}, \{1\} \rangle$, where $\underline{\mathbf{2}} = \langle 2, \wedge, \vee \rangle$ is the two-element distributive lattice. We completely characterize the class of reduced matrices for this deductive system, disproving the common belief that their algebraic reducts are all distributive lattices. In fact, that class is a proper subclass of distributive lattices with unit and is not even a quasivariety.

Let us begin by recalling some basic definitions we will use. Throughout this note, all algebras are of type (2,2).

– A *logical matrix* for $\vdash$ is a pair $\langle \underline{\mathbf{A}}, F \rangle$ where $\underline{\mathbf{A}} = \langle A, \wedge, \vee \rangle$ is an algebra and $F \subseteq A$ is such that if $\Gamma \vdash \varphi$ and $h \in Hom(\underline{\mathbf{F}}, \underline{\mathbf{A}})$ then $h(\Gamma) \subseteq F$ implies $h(\varphi) \in F$.

– Given an algebra $\underline{\mathbf{A}}$, the *Leibniz operator* $\Omega_{\underline{\mathbf{A}}}$ assigns to each $F \subseteq A$ the largest congruence of $\underline{\mathbf{A}}$ compatible with F, i.e. such that F is the union of equivalence classes. It is easy to see (cf. [1, p. 11]) that

$$\Omega_{\underline{\mathbf{A}}}(F) = \{(a,b) : \ \varphi^{\underline{\mathbf{A}}}(a, c_1, \ldots, c_n) \in F \Leftrightarrow \varphi^{\underline{\mathbf{A}}}(b, c_1, \ldots, c_n) \in F \text{ for all } \varphi(v_0, v_1, \ldots, v_n) \in Form \text{ and } c_1, \ldots, c_n \in A\}$$

– A matrix $\langle \underline{\mathbf{A}}, F \rangle$ is said to be *reduced* (or also *simple*) if $\Omega_{\underline{\mathbf{A}}}(F) = \Delta_A$, the identity congruence of $\underline{\mathbf{A}}$.

Reduced matrices play an important role in algebraic logic. If $\mathcal{L} = \langle \underline{\mathbf{F}}, \vdash \rangle$ is a deductive system, then the class $\mathcal{R}$ of reduced matrices for $\mathcal{L}$

forms a matrix semantics of $\mathcal{L}$, i.e. for any $\Gamma \cup \{\varphi\} \subseteq Form$, we have $\Gamma \vdash \varphi \Leftrightarrow \Gamma \models_{\mathcal{R}} \varphi$. In particular, if $\mathcal{L}$ is *algebraizable* [1], the class of algebraic reducts of $\mathcal{R}$ is the equivalent quasivariety semantics of $\mathcal{L}$ (cf. [1, cor. 5.3]), that is, the quasivariety canonically associated with $\mathcal{L}$. In the case that concerns us, $\vdash$ is not algebraizable, not even protoalgebraic (cf. [3, prop. 2.8]), and reduced matrices do not seem to play a significant role in the algebraic study of $\mathcal{L}$.

In [3] a Gentzen-style presentation is given for $\mathcal{L}$. Using the concept of "model of a Gentzen calculus" a close connection is established between the variety ID of distributive lattices and $\mathcal{L}$. It results that distributive lattices are the algebraic reducts of these reduced models. This connection however, depends on the Gentzen calculus and not just on $\mathcal{L}$.

In [2] a Hilbert-style presentation is given for $\mathcal{L}$. It has no axioms, and has the following rules of inference:

(R1) $\varphi \wedge \psi \vdash \varphi$	(R7) $\varphi \vee (\psi \vee \xi) \vdash (\varphi \vee \psi) \vee \xi$
(R2) $\varphi \wedge \psi \vdash \psi \wedge \varphi$	(R8) $(\varphi \vee \psi) \vee \xi \vdash \varphi \vee (\psi \vee \xi)$
(R3) $\{\varphi, \psi\} \vdash \varphi \wedge \psi$	(R9) $\varphi \vee (\psi \wedge \xi) \vdash (\varphi \vee \psi) \wedge (\varphi \vee \xi)$
(R4) $\varphi \vdash \varphi \vee \psi$	(R10) $(\varphi \vee \psi) \wedge (\varphi \vee \xi) \vdash \varphi \vee (\psi \wedge \xi)$
(R5) $\varphi \vee \psi \vdash \psi \vee \varphi$	(R11) $\varphi \wedge (\psi \vee \xi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \xi)$
(R6) $\varphi \vee (\varphi \vee \psi) \vdash \varphi \vee \psi$	(R12) $\varphi \vee \varphi \vdash \varphi$

We remark that rules (R6), (R8) and (R11) are derivable from the rest:

(R8):

1)	$(\varphi \vee \psi) \vee \xi$	Hypothesis
2)	$\xi \vee (\varphi \vee \psi)$	1),(R5)
3)	$(\xi \vee \varphi) \vee \psi$	2),(R7)
4)	$\psi \vee (\xi \vee \varphi)$	3),(R5)
5)	$(\psi \vee \xi) \vee \varphi$	4),(R7)
6)	$\varphi \vee (\psi \vee \xi)$	5),(R5)

(R6):

1)	$\varphi \vee (\varphi \vee \psi)$	Hypothesis
2)	$(\varphi \vee \psi) \vee \varphi$	1),(R5)
3)	$((\varphi \vee \psi) \vee \varphi) \vee \psi$	2),(R4)
4)	$(\varphi \vee \psi) \vee (\varphi \vee \psi)$	3),(R8)
5)	$\varphi \vee \psi$	4),(R12)

(R11):

1)	$\varphi \wedge (\psi \vee \xi)$	Hypothesis	6)	$(\xi \vee \varphi) \wedge (\xi \vee \psi)$	5),4,(R3)
2)	φ	1),(R1)	7)	$\xi \vee (\varphi \wedge \psi)$	6),(R10)
3)	$\psi \vee \xi$	1),(R2),(R1)	8)	$(\varphi \wedge \psi) \vee \xi$	7),(R5)
4)	$\xi \vee \psi$	3),(R5)	9)	$(\varphi \wedge \psi) \vee \varphi$	2),(R4),(R5)
5)	$\xi \vee \varphi$	2),(R4),(R5)	10)	$(\varphi \wedge \psi) \vee (\varphi \wedge \xi)$	9),8),(R3),(R10)

Hilbert-style presentations of logics are useful when working with their matrices. In our case we obtain a particularly simple characterization of the Leibniz operator on matrices of $\mathcal{L}$:

PROPOSITION. *Let $\langle \underline{\mathbf{A}}, F \rangle$ be a matrix for $\mathcal{L}$, and $a, b \in A$. Then $(a, b) \in \Omega_{\underline{\mathbf{A}}}(F)$ if and only if*

$$\forall c \in A, \quad a \vee c \in F \Longleftrightarrow b \vee c \in F. \tag{1}$$

PROOF: $(\Rightarrow)$ This a particular case of the general characterization of $\Omega_{\underline{\mathbf{A}}}(F)$, when φ is $v_0 \vee v_1$.
$(\Leftarrow)$ Suppose (1) holds and $(a, b) \notin \Omega_{\underline{\mathbf{A}}}(F)$. Then there is $\varphi(v_0, \ldots, v_n) \in$ *Form* with the property.

$$\exists c_1, \ldots, c_n \in A \text{ with } \varphi^{\underline{\mathbf{A}}}(a, c_1, \ldots, c_n) \in F \text{ but } \varphi^{\underline{\mathbf{A}}}(b, c_1, \ldots, c_n) \notin F. \tag{2}$$

Observe that any formula ψ in the same variables and such that $\varphi \vdash\dashv \psi$ also has property (2). Choose $\varphi(v_0, \ldots, v_n)$ of least "complexity" (number of $\wedge$ and $\vee$) satisfying (2). Clearly v_0 must appear in φ, but φ cannot be v_0: if that were the case, then $a \in F$ and $b \notin F$. By (R4) $a \vee b \in F$, and by (R12) $b \vee b \notin F$. But this contradicts (1). φ cannot be a conjunction either, because if $\varphi = \varphi_1 \wedge \varphi_2$ then (R1), (R2) and (R3) would prove that either φ_1 or φ_2 satisfy (2), contradicting the choice of φ with least complexity. Thus φ must be a disjunction, and we can apply Lemma 3 of [2] to find formulas $\varphi_1, \ldots, \varphi_k$ in the same variables, such that none of them is a disjunction, $\varphi \vdash\dashv \varphi_{\sigma(1)} \vee (\varphi_{\sigma(2)} \vee (\ldots \vee (\varphi_{\sigma(k-1)} \vee \varphi_{\sigma(k)}) \ldots))$, for every permutation σ, and the complexity of the right-hand-side formula is the same as that of φ. Therefore each of these right-hand-side formulas satisfies (2) with least complexity. From this and (R1), (R2), (R3), (R9) and (R10) it follows that none of the φ_i can be a conjunction, and therefore it must be a variable. Finally, the least complexity condition forces $k = 2$, that is $\varphi \vdash\dashv v_0 \vee v_1$, which satisfies (2). But this contradicts (1). □

This Proposition allows us to obtain the following characterization of the reduced matrices for $\mathcal{L}$.

THEOREM. *$\langle \underline{\mathbf{A}}, F\rangle$ is a reduced matrix for $\mathcal{L}$ if and only if $\underline{\mathbf{A}}$ is a distributive lattice with unit 1, satisfying:*

$$\forall a, b \in A, \text{ if } a < b \text{ then } \exists c \in A \text{ with } a \vee c \neq 1 \text{ and } b \vee c = 1 \tag{3}$$

and $F = \{1\}$, or else $A = \{1\}$ and $F = \emptyset$.

PROOF. ($\Rightarrow$) Let $\varphi \approx \psi$ be any equation valid in ID, the variety of distributive lattices, and v a variable not appearing in $\varphi \approx \psi$. Then the equation $\varphi \vee v \approx \psi \vee v$ is also valid in ID. By the Completeness Theorem in [2], we have $\varphi \vee v \dashv\vdash \psi \vee v$. Since $\langle \underline{\mathbf{A}}, F\rangle$ is a matrix for $\mathcal{L}$ then for any interpretation $\overline{a}$ in $\underline{\mathbf{A}}$ and any $c \in A$, we have $\varphi^{\underline{\mathbf{A}}}(\overline{a}) \vee c \in F$ iff $\psi^{\underline{\mathbf{A}}}(\overline{a}) \vee c \in F$. By the Proposition this implies that $(\varphi^{\underline{\mathbf{A}}}(\overline{a}), \psi^{\underline{\mathbf{A}}}(\overline{a})) \in \Omega_{\underline{\mathbf{A}}}(F)$, and since $\langle \underline{\mathbf{A}}, F\rangle$ is reduced, $\varphi^{\underline{\mathbf{A}}}(\overline{a}) = \psi^{\underline{\mathbf{A}}}(\overline{a})$, that is the equation $\varphi \approx \psi$ holds in $\underline{\mathbf{A}}$. Thus $\underline{\mathbf{A}} \in ID$. If $F = \emptyset, A$ must be a singleton for the matrix to be reduced. On the other hand if $F \neq \emptyset$ and $a, b \in F$, then for any $c \in A$, by (R4), we have $a \vee c, b \vee c \in F$, which by the same argument as before implies $a = b$. Thus $F = \{1\}$ for some $1 \in A$, and (R4) again implies that it is the maximum. Finally, if $a < b$, then $(a, b) \notin \Omega_{\underline{\mathbf{A}}}(\{1\})$ and the Proposition implies (3).
($\Leftarrow$) If $\underline{\mathbf{A}} \in ID$ with unit 1 satisfies (3), it is easy to check that $\langle \underline{\mathbf{A}}, \{1\}\rangle$ is a matrix for $\mathcal{L}$, by using the Hilbert-style presentation of $\mathcal{L}$ given above, and condition (3) with the Proposition imply that it is reduced. Finally, $\langle \{1\}, \emptyset\rangle$ is also a reduced matrix for $\mathcal{L}$. □

The above Theorem disproves the common belief that the class of algebraic reducts of reduced matrices for $\mathcal{L}$ consists of all distributive lattices. This assertion appears without proof in [4], page 162. For example, the only chain in this class is $\underline{\mathbf{2}}$, whereas any complemented distributive lattice is in it, since we can take c to be the complement of b in (3). Therefore this class is not closed under the subalgebra operator, and thus it is not a quasivariety.

References

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