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A RELATION BETWEEN NAMESAKES IN MODAL LOGIC

Abstract

The closure under a certain rule of the modal system usually called "$K4$" is shown to be the modal system called "$K4$" by Sobociński.

A modal system S provides double cancellation iff for all sentences A and B, if $\vdash_S LA \equiv LB$ and $\vdash_S MA \equiv MB$ then $\vdash_S A \equiv B$. It is of course not required that $\vdash_S ((LA \equiv LB)\&(MA \equiv MB)) \supset (A \equiv B)$. If S is any modal system, there is a smallest normal extension $S+$ of S providing double cancellation (terminology as in [1]). For reasons given in [3], the double cancellation rule may be of some philosophical interest. This note extends the results of [3] and [4] on double cancellation in normal systems.

The name "$K4$" is ambiguous. It is now customarily used for the smallest normal system containing the schema $LA \supset LLA$; this practice is followed here. Sobociński used the same name for the smallest system containing $KT4$ ($= S4$) and the schemata $A \supset (MLA \supset LA)$ and $MLA \equiv LMA$ ([2]); this normal system is here referred to as $K4_{Sob}$. The relation to be established between these namesakes is that $K4+ = K4_{Sob}$; it will also be shown that $K4_{Sob}$ is the only normal extension of $K4$ providing double cancellation in which there is not modal collapse. This extends the results of [3], in which it was shown that $KD4!+ = K4_{Sob}$, where $KD4!$ is the smallest normal system containing the formula MT (T being a constant tautology) and the schema $LLA \equiv LA$, and [4], in which it is shown that $KW+$ is inconsistent, where KW is "provability logic", the smallest normal system containing $L(LA \supset A) \supset LA$, an extension of $K4$.

Lemma A. $\vdash_{K4+} (A\&M(A\&LA)) \supset LA$.

Proof. For any formula A, the following are theorems of $K4+$ (where on the right a subsystem of $K4+$ is given in any normal extension of which the reasoning can be carried out; e.g. "K" means that the reasoning can be carried out in any normal system):

(1) $(A\&(LA\vee \sim M(A\&LA))) \supset A$ — PC

(2) $L(A\&(LA\vee \sim M(A\&LA))) \supset LA$ — $1, K$

(3) $M(A\&(LA\vee \sim M(A\&LA))) \supset MA$ — $1, K$

(4) $LA \supset (LA\vee \sim M(A\&LA))$ — PC

(5) $LLA \supset L(LA\vee \sim M(A\&LA))$ — $4, K$

(6) $LA \supset LLA$ — $K4$

(7) $LA \supset L(LA\vee \sim M(A\&LA))$ — $5, 6, PC$

(8) $LA \supset L(A\&(LA\vee \sim M(A\&LA)))$ — $7, K$

(9) $LA \equiv L(A\&(LA\vee \sim M(A\&LA)))$ — $2, 8, PC$

(10) $A \supset (M(A\&LA) \vee (A\&(LA\vee \sim M(A\&LA))))$ — PC

(11) $MA \supset (MM(A\&LA) \vee M(A\&(LA\vee \sim M(A\&LA))))$ — $10, K$

(12) $MM(A\&LA) \supset M(A\&LA)$ — $K4$

(13) $(A\&LA) \supset (A\&(LA\vee \sim M(A\&LA)))$ — PC

(14) $M(A\&LA) \supset M(A\&(LA\vee \sim M(A\&LA)))$ — $13, K$

(15) $MM(A\&LA) \supset M(A\&(LA\vee \sim M(A\&LA)))$ — $12, 14, K$

(16) $MA \supset M(A\&(LA\vee \sim M(A\&LA)))$ — $11, 15, K$

(17) $MA \equiv M(A\&(LA\vee \sim M(A\&LA)))$ — $3, 16, PC$

(18) $A \equiv (A\&(LA\vee \sim M(A\&LA)))$ — $9, 17, K4+$

(19) $(A\&M(A\&LA)) \supset LA$ — $18, PC$

LEMMA B. $\vdash_{K4+} (A\&LMA) \supset LA$.

PROOF. The following are theorems of $K4+$:

(1) $L(A\&M(\sim A\&MA)) \supset (LA\&LM(\sim A\&MA))$ — K

(2) $LA \supset\sim M(\sim A\&MA)$ — K

(3) $(LA\&M(\sim A\&MA)) \supset (\sim A\&M(A\&M \sim A))$ — $2, PC$

(4) $(LLA\&LM(\sim A\&MA)) \supset L(\sim A\&M(A\&M \sim A))$ — $3, K$

(5) $(LA\&LM(\sim A\&MA)) \supset L(\sim A\&M(A\&M \sim A))$ — $4, K4$

(6) $L(A\&M(\sim A\&MA)) \supset L(\sim A\&M(A\&M \sim A))$ — $1, 5, PC$

(7) $L(\sim A\&M(A\&M \sim A)) \supset L(A\&M(\sim A\&MA))$ — Like 6

(8) $L(A\&M(\sim A\&MA)) \equiv L(\sim A\&M(A\&M \sim A))$ — $6, 7, PC$

(9) $(A\& \sim LA) \supset\sim M(A\&LA)$ — Lemma A

(10) $(A\&M \sim A) \supset L(\sim A \vee M \sim A)$ — $9, K$

(11) $(A\&M \sim A) \supset LL(\sim A \vee M \sim A)$ — $10, K4$

(12) $M(\sim A\&MA) \supset M \sim A$ — K

(13) $(A\&M(\sim A\&MA)) \supset LL(\sim A \vee M \sim A)$ — $11, 12, PC$

(14) $(M(\sim A\&MA)\&LL(\sim A \vee M \sim A)) \supset$
$M(\sim A\&MA\&L(\sim A \vee M \sim A))$ — K

(15) $(\sim A\&MA)\&L(\sim A \vee M \sim A) \supset$
$(\sim A\&M(A\&M \sim A))$ — K

(16) $M(\sim A\&MA\&L(\sim A \vee M \sim A)) \supset$
$M(\sim A\&M(A\&M \sim A))$ — $15, K$

(17) $(M(\sim A\&MA)\&LL(\sim A \vee M \sim A)) \supset$
$M(\sim A\&M(A\&M \sim A))$ — $14, 15, PC$

(18) $(A\&M(\sim A\&MA)) \supset M(\sim A\&M(A\&M \sim A))$ — $13, 17, PC$

(19)	$M(A\&M(\sim A\&MA)) \supset MM(\sim A\&M(A\&M \sim A))$	$18, K$
(20)	$M(A\&M(\sim A\&MA)) \supset M(\sim A\&M(A\&M \sim A))$	$19, K4$
(21)	$M(\sim A\&M(A\&M \sim A)) \supset M(A\&M(\sim A\&MA))$	Like 20
(22)	$M(A\&M(\sim A\&MA)) \equiv M(\sim A\&M(A\&M \sim A))$	$20, 21, PC$
(23)	$(A\&M(\sim A\&MA)) \equiv (\sim A\&M(A\&M \sim A))$	$8, 21, K4+$
(24)	$\sim (A\&M(\sim A\&MA))$	$23, PC$
(25)	$((\sim M \sim A\&MA)\&LMA) \supset LA$	K
(26)	$(A\&LMA) \supset LA$	$24, 25, PC$

Lemma C. $\vdash_{K4+} LLA \equiv LA$.

Proof. The following are theorems of $K4+$:

(1)	$(LA\& \sim LA) \supset A$	K
(2)	$(LLA\&L \sim LA) \supset LA$	$1, K$
(3)	$LLA \supset (MLA \vee LA)$	$2, K$
(4)	$MLLA \supset (MMLA \vee MLA)$	$3, K$
(5)	$MMLA \supset MLA$	$K4$
(6)	$MLLA \supset MLA$	$4, 5, PC$
(7)	$LA \supset LLA$	$K4$
(8)	$MLA \supset MLLA$	$7, K$
(9)	$MLLA \equiv MLA$	$6, 8, PC$
(10)	$((LA\vee \sim LLA)\&LM(LA\vee \sim LLA)) \supset L(LA\vee \sim LLA)$	Lemma B
(11)	$((LA\vee \sim LLA)\&LLA) \supset LA$	PC
(12)	$(L(LA\vee \sim LLA)\&LLLA) \supset LLA$	$11, K$
(13)	$(LM(LA\vee \sim LLA)\&LLLA) \supset LLA$	$10, 12, PC$

(14) $((LA\vee \sim LLA)\&M((LA\vee \sim LLA)\&L(LA\vee \sim LLA))) \supset$
$L(LA\vee \sim LLA)$ Lemma A

(15) $(M((LA\vee \sim LLA)\&L(LA\vee \sim LLA))\&LLLA) \supset LLA$ $12, 14, PC$

(16) $MLA \supset M(LA\vee \sim LLA)$ K

(17) $LMLA \supset LM(LA\vee \sim LLA)$ $16, K$

(18) $(LMLA\&LLLA) \supset LLA$ $13, 17, PC$

(19) $(\sim LA\&LA) \supset (A\&(LA\vee \sim LLA))$ PC

(20) $(\sim MLA\&LLA) \supset (LA\&L(LA\vee \sim LLA))$ $19, K$

(21) $(\sim MLA\&LLA) \supset ((LA\vee \sim LLA)\&L(LA\vee \sim LLA))$ $20, PC$

(22) $M(\sim MLA\&LLA) \supset$
$M((LA\vee \sim LLA)\&L(LA\vee \sim LLA))$ $21, K$

(23) $(M(\sim MLA\&LLA)\&LLLA) \supset LLA$ $15, 22, PC$

(24) $(\sim LMLA\&LLLA) \supset M(\sim MLA\&LLA)$ K

(25) $(\sim LMLA\&LLLA) \supset LLA$ $23, 24, PC$

(26) $LLLA \supset LLA$ $18, 25, PC$

(27) $LLA \supset LLLA$ $K4$

(28) $LLLA \equiv LLA$ $26, 27, PC$

(29) $LLA \equiv LA$ $9, 28, K4+$

Lemma D. $\vdash_{K4+} MT$.

Proof. The following are theorems of $K4+$:

(1) $LLMT \equiv LMT$ Lemma C

(2) $LL \sim T \equiv L \sim T$ Lemma C

(3) $MMT \equiv MT$ $2, K$

(4) $MLMT \supset MT$ K

(5) $MLMT \supset MMT$ $3, 4, PC$

(6) $L \sim T \supset LMT$ K

(7) $ML \sim T \supset MLMT$ $6, K$

(8) $(MMT\&LLMT) \supset MLMT$ K

(9) $(MMT\&LMT) \supset MLMT$ $8, K4$

(10) $(MMT\& \sim ML \sim T) \supset MLMT$ $9, K$

(11) $MMT \supset MLMT$ $7, 10, PC$

(12) $MLMT \equiv MMT$ $5, 11, PC$

(13) $LMT \equiv MT$ $1, 12, K4+$

(14) $\sim MT \supset LMT$ K

(15) MT $13, 14, PC$

LEMMA E. $K4+ = K4_{Sob}$.

PROOF. It is shown in [3] that $KD4!+ = K4_{Sob}$. By Lemmas C and D, $KD4! \subseteq K4+$, so $KD4!+ \subseteq K4++ = K4+$. But $K4 \subseteq KD4!$, so $K4+ \subseteq KD4!+$.

LEMMA F. *The only normal proper extensions of $K4_{Sob}$ are Triv (K plus $A \equiv LA$) and the inconsistent system.*

PROOF. Let S be any normal extension of $K4_{Sob}$. By results noted in [3], the frame of any generated submodel of the canonical model of $K4_{Sob}$ consists of either a single reflexive point or two non-simmetrically related reflexive points, and $K4_{Sob}$ is valid on all and only such frames. Since any non-theorem of S is falsified in a generated submodel of the canonical model of $K4_{Sob}$ that is a model of S, S has the finite model property, and so by Corollary 8.14 of [1] is characterized by a class C of frames, each of must be of one of the two mentioned kinds (else it would not be a frame for $K4_{Sob}$, and so not for S). If C contains a two-point frame, $S = K4_{Sob}$.

If C is non-empty but contains only one-point frames, $S = Triv$. If C is empty, S is inconsistent.

THEOREM. *The normal extensions of $K4$ providing double cancellation are $K4_{Sob}$, $Triv$ and the inconsistent system.*

PROOF. Any normal extension of $K4$ providing double cancellation is one of the three mentioned systems by Lemmas E and F. $K4_{Sob}$ provides double cancellation by Lemma E. $Triv$ and the inconsistent system provide it trivially.

References

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