

Jacek Malinowski

QUASIVARIETIES OF MODULAR ORTHOLATTICES

Abstract

Lattice of subquasivarieties of variety generated by modular ortholattices MO_n , $n \in \omega$ and $MO\omega$ is described.

In [3] Igošin has proved that any subquasivariety of the variety \mathbf{M}_ω is a variety. \mathbf{M}_ω denoted variety generated by modular lattice M_ω , where M_ω is the lattice of length two with ω atoms. For short proof of this fact see [2]. In this note we present an orthomodular counterpart of this result. In the case of modular ortholattices not every subquasivariety of the variety $\mathbf{MO}\omega$ forms a variety. We give however, a complete description of the lattice of subquasivariety of $\mathbf{MO}\omega$. This description is a consequence of some known through yet not published general results, summarized here in lemmas 1, 2, 3. For basic fact from universal algebra we refer to [1].

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Let \mathbf{K} be a class of algebras, by $Q(\mathbf{K})$ we denote the least quasivariety containing the class \mathbf{K} . For every class \mathbf{K} of algebras an algebra $A \in \mathbf{K}$ is *subdirectly \mathbf{K} -irreducible* iff for every set $(A_i : i \in I)$ of algebras from \mathbf{K} if A is a subdirect product of $(A_i : i \in I)$ then A is isomorphic to A_{i_0} for some $i_0 \in I$. If \mathbf{K} is a quasivariety then every algebra $A \in \mathbf{K}$ is isomorphic to subdirect product of subdirectly \mathbf{K} -irreducible algebras. A non trivial algebra A will be called *critical* iff A is subdirectly $Q(A)$ -irreducible. A finite algebra is critical if and only if it does not belong to the quasivariety generated by its proper subalgebras. Every quasivariety of

algebras is generated by its finitely generated critical algebras. A class \mathbf{K} of algebras will be called *locally finite* if and only if every finitely generated algebra from \mathbf{K} is finite. If a class \mathbf{K} is generated by a finite set of finite algebras then \mathbf{K} will be called *finitely generated*. By $\text{Con}(A)$ we shall denote the lattice of congruences of algebra A . If for every algebra $A \in \mathbf{K}$ the lattice $\text{Con}(A)$ is distributive then the class \mathbf{K} will be called *congruence distributive*. An algebra A with two congruences only (identity and full congruence) will be called *simple*. Variety \mathbf{K} will be called *semi-simple* if and only if every subdirectly irreducible algebra from the class \mathbf{K} is simple. Let $\theta_1, \theta_2 \in \text{Con}(A)$ for some algebra A , we will define the relation $\theta_1 \circ \theta_2$ on A in the following manner: $a(\theta_1 \circ \theta_2)b$ iff there is $c \in A$ such that $a\theta_1 c$ and $c\theta_2 b$. A class \mathbf{K} of algebras will be called *congruence permutable* if and only if for every $A \in \mathbf{K}$ and $\theta_1, \theta_2 \in \text{Con}(A)$ following equality holds $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1 = \theta_1 \vee \theta_2$.

LEMMA 1. *Let \mathbf{K} be a semi-simple, congruence permutable variety of algebras. Every finite critical algebra $A \in \mathbf{K}$ is isomorphic to a direct product of finitely many pairwise non-isomorphic simple algebras from \mathbf{K} .*

PROOF. A is a subdirect product of subdirectly irreducible algebras $A/\theta_1, \dots, A/\theta_n$ for some $\theta_1, \dots, \theta_n \in \text{Con}(A)$. We can assume that the set $\theta_1, \dots, \theta_n$ is minimal in the sense that A is isomorphic with no proper subset of the set $\{A/\theta_1, \dots, A/\theta_n\}$. It is easy to verify that congruences $\theta_1, \dots, \theta_n$ are pairwise incomparable, algebras $A/\theta_1, \dots, A/\theta_n$ are pairwise non-isomorphic, no congruences $\theta_1, \dots, \theta_n$ are trivial.

We can assume that $n = 2$, the proof for $n \in \omega$ is similar. It is enough to show that the embedding

$$a \mapsto ([a]\theta_1, [a]\theta_2)$$

is “onto”. Let $[a_1]\theta_1 \in A/\theta_1, [a_2]\theta_2 \in A/\theta_2$. Since θ_1 and θ_2 are incomparable, hence $\theta_1 \vee \theta_2 > \theta_1$, since A/θ_1 is simple hence $\theta_1 \vee \theta_2$ is the full congruence. Since \mathbf{K} is congruence permutable so $\theta_1 \vee \theta_2 = \theta_1 \circ \theta_2$, this means that $\theta_1 \circ \theta_2$ is the full congruence on A , hence there exists $a \in A$ such that $a_1\theta_1 a\theta_2 a_2$ so $[a_1]\theta_1 = [a]\theta_1$ and $[a_2]\theta_2 = [a]\theta_2$. \square

LEMMA 2. *If $A = A_1 \times \dots \times A_n$ is finite algebra, and there exist $i, j, k \in \{1, \dots, n\}$ such that $A_i \hookrightarrow A_j, A_j \hookrightarrow A_k$, (symbol \hookrightarrow denote “is isomorphic with a subalgebra of”) then A is not critical.*

PROOF. Suppose that $A_1 \hookrightarrow A_2 \hookrightarrow A_3$, let B_1 be the algebra $A_1 \times A_3$, let B_2 be the algebra $A_1 \times A_2$, then $A \hookrightarrow B_1 \times B_2$ and B_1, B_2 are isomorphic with proper subalgebras of $A_1 \times A_2 \times A_3$. \square

LEMMA 3. *If the algebras A, B are non-isomorphic, finite, simple and do not contain one-element subalgebras then $A \times B$ is critical.*

PROOF. If $A_1 \times B_1 \hookrightarrow A \times B$, A_1, B_1 are non-isomorphic and simple then it is easy to check $A_1 \hookrightarrow A$ and $B_1 \hookrightarrow B$. An algebra $A \times B$ cannot be isomorphic to subdirect product of such algebras. \square

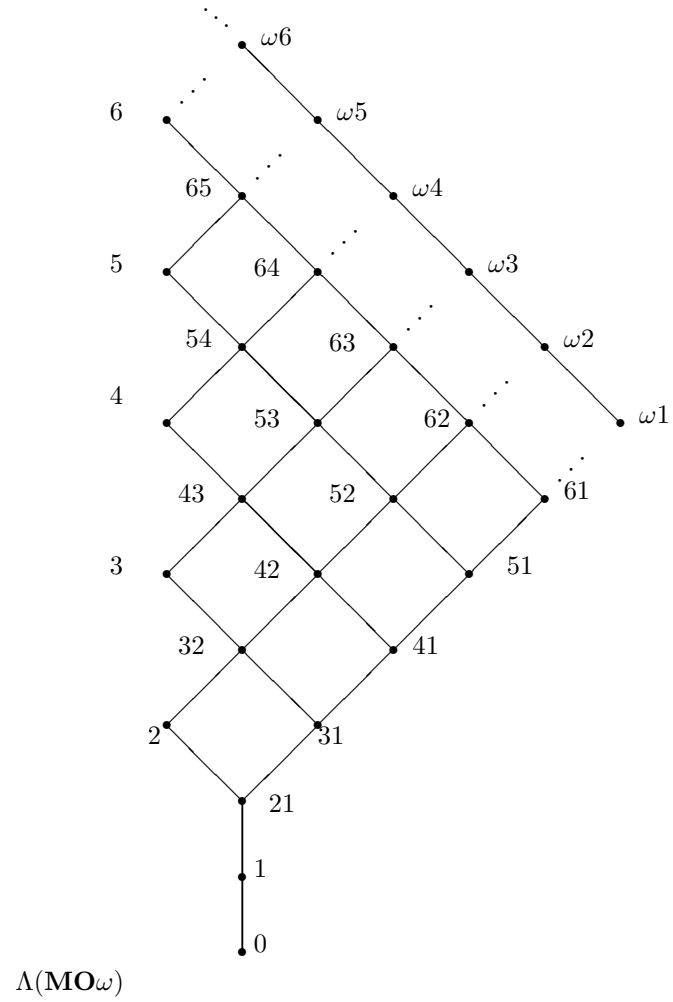
An algebra $A = (A, \vee, \wedge, ')$ of the type $(2, 2, 1)$ will be called an *ortholattice* iff (A, \vee, \wedge) is a bounded lattice, $'$ is anti-monotone complementation on A . An ortholattice which satisfies equality: $x \vee (x' \wedge (x \vee y)) = x \wedge y$ will be called an *orthomodular lattice*. The variety of all orthomodular ortholattices will be denoted by *OML*. An ortholattice is modular (MOL for short) iff for $x \leq y$, $x \vee (y \wedge z) = y \wedge (x \vee z)$. Let MO_n for $n \in \omega$ and MO_ω denote the modular ortholattice with $2n$ (ω respectively) pairwise incomparable elements and the bounds. Varieties generated by ortholattices MO_n and MO_ω will be denoted by \mathbf{MO}_n and \mathbf{MO}_ω respectively. Roddy in [5] has proved that these varieties form an initial chain in lattice of varieties of orthomodular lattices. The variety *OML* is semi-simple (see [4], p. 79). An obvious consequence of these two facts is that the only finite simple MOL are MO_0 (one element algebra), MO_1 (two element Boolean algebra), MO_2, \dots, MO_n, \dots . Moreover the only locally finite variety of MOL are $\mathbf{MO}_0, \mathbf{MO}_1, \mathbf{MO}_2, \dots, \mathbf{MO}_n, \dots$, and \mathbf{MO}_ω .

The ternary polynomial $p(x, y, z) = f(z, f(x, y, z), x)$ where $f(x, y, z) = (z \wedge y) \vee (y' \wedge (x \vee y))$ meets in *OML* the assumption of Malcev theorem. Hence the variety *OML* is congruence permutable.

It is easy to see that for finitely generated variety $\mathbf{K} \in \mathbf{OML}$, the lattice $\Lambda(\mathbf{K})$ is finite.

Directly from lemmas 1, 2, 3 we obtain that the only finite critical algebras in \mathbf{MO}_ω are the algebras $MO_k \times MO_l$ where $k, l \in 1, 2, \dots, n$ and $k < l$. Below we present the lattice $\Lambda(\mathbf{MO}_\omega)$ of subquasivarieties of the variety \mathbf{MO}_ω . The numbers $0, 1, 2, \dots, (nk)$ denote the quasivarieties $Q(MO_0)$ (-the trivial variety), $Q(MO_1), Q(MO_2), \dots, Q(MO_n \times MO_k)$ respectively). For any quasivariety $q \in \Lambda(\mathbf{MO}_\omega)$ the interval $[0, q]$ is iso-

morphic to the lattice $\Lambda(\mathbf{MOn})$ of subquasivarieties of the variety \mathbf{MOn} . The lattice $\Lambda(\mathbf{MOn})$ has $\frac{n(n+1)}{2}$ elements.



References

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Section of Logic
Institute of Philosophy and Sociology
Polish Academy of Sciences
Poland