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CONTINUOUS OPERATIONS ON SPACES OF STRUCTURES

Given a model theoretic logic L. The (large) topological space $E_\tau(L)$ of τ-structures topologized by the L-elementary classes is completely regular and uniformizable by a standard topological argument. In fact, it has a canonical totally bounded uniformity that faithfully reflects the properties of L.

The cartesian product $E_\tau \times E_\rho = \{[A, B] : A \in E_\tau, B \in E_\rho\}$ has a natural product uniformity, but inherits also a uniformity as subspace of $E_\mu(L)$ for an appropriate $\mu \supseteq \tau, \rho$. It may be shown that the second uniformity is finer than the first if and only if L has relativizations and both uniformities coincide if and only if L has, in addition the uniform reduction property for pairs (in the compact case just the Feferman-Vaught property for pairs). A similar result holds for infinite products. We exploit this fact plus the fact that in theses spaces compactness is equivalent to Cauchy-completeness to generalize a theorem of Mundici [Mu] characterizing compactness, and show also that under reduction for infinite disjoint sums compactness is equivalent to normality of the model spaces.

A partial operation $F : E_\tau(L) \to E_\rho(M)$ is uniformly continuous for the topologies induced by the logic L, M if and only if it has the *uniform reduction property* in the senses of Gaifman and Feferman, see [Ma] or, equivalently it is a *construction for a translation* in the sense of Szczerba and Krynicki. Call an operation $L - PC$ (respectively $L - RPC$) if its graph in $E_\tau \times E_\rho$ is PC in L (resp. RPC in L). We show that for compact L an $L - RPC$ operation is uniformly continuous if and only if it is so for infinite structures, if and only if it preserves elementary equivalence. Let $Int(L, M)$ and $\Delta Int(L, M)$ mean respectively that M allows interpolation of relativized PC classes by $Int_R(L,M)$ and $\Delta Int_R(L,M)$, respectively. Then, assuming closure under relativizations of the logics we have:

(i) $Int_R(L,M)$ (resp. $\Delta Int_R(L,M)$) if and only if any partial (resp. total) $L - RPC$ operation $F : E_\tau(M) \to E_\rho(L)$ is uniformly continuous.

(ii) $Int(L,M)$ (resp. $\Delta Int(L,K)$) if and only if any partial (resp. total) $L - PC$ operation satisfying $|F(A)| \leq |A|$ is uniformly continuous.

We have analogous results without assuming closure under relativizations. There are also topological characterizations of Beth's definability theorem. Using them we show some theorems on the existence of models with infinite groups of automorphisms. In particular, a strong form of Shelah's homogeneity property [Ma] is shown to be equivalent to Robinson's lemma, in the presence of uniform reduction for pairs.

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