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Lauri Hella and Kerkko Luosto

FINITE GENERATION AND n-ARY QUANTIFIERS

This paper is a survery on the technique to prove logics non-finitely generated originated in [H] and later used in [HL] and [HK]. The basic idea is that many $(n+1)$-ary quantifiers Q are non-redundant in the sense that if $\mathcal{Q}$ is a set of n-ary quantifiers, then Q is not definable in $\mathcal{L}_{\omega\omega}(\mathcal{Q})$ (or even in $\mathcal{L}_{\infty\omega}(\mathcal{Q})$). Here, quantifier Q' is n-ary, if $ar(Q') \leq n$ where the arity $ar(Q') = sup\{n_R : R \in \tau\}$ is the supremum of the numbers of variables in formulas bounded by the quantifier Q'. We say that the *strict (strong) dimension* of the quantifier Q above is $n+1$ (similarly for model-classes). In practice, a back-and-forth characterization and a specific construction are needed to prove the non-redundancy. Back-and-forth systems are developed for logics $\mathcal{L}_{\infty\omega}(\mathbf{Q}_n)$ and $\mathcal{L}_{\infty\omega}(\mathbf{Q}_G)$ where $\mathbf{Q}_n$ is a collection of n-ary quantifiers and $\mathbf{Q}_G$ a subcollection of $\mathbf{Q}_n$ satisfying certain symmetricity conditions (G is a permutation group). Non-trivial pairs of $\mathcal{L}_{\infty\omega}(\mathbf{Q}_n)$-equivalent models are then constructed by means of a game.

As applications, some logics $\mathcal{L}$ can be shown to be strongly hierarchical, by which we mean that the dimensions of elementary classes of $\mathcal{L}$ are not bounded by any $n < \omega$. Especially these logics are not finitely generated. These examples fall into two main categories. On one hand, some logics like Magidor-Malitz logics $\mathcal{L}_{\omega\omega}(Q^n_\alpha)_{n<\omega}$ (for $\alpha > 0$) have an obvious hierarchy. On the other hand, various closures of finitely generated logics are strongly hierarchical, for instance the Beth closures $B(\mathcal{L}_{\omega\omega}(Q_\alpha)), B(\mathcal{L}_{\omega\omega}(Q^{cf}_C))$ and the Δ-closure $\Delta(\mathcal{L}_{\omega\omega}(Q_\alpha, Q_{\alpha+1}))$ for $\aleph_\alpha$ uncountable and regular and C untrivial.

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Department of Mathematics
University of Helsinki
Hallituskatu 15
SF 00100 Helsinki, Finland