Bulletin of the **S**ection of **L**ogic
Volume 2/1 (1973), pp. 3–6
reedition 2013 [original edition, pp. 3–7]

Viktor K. Finn

A CRITERION OF FUNCTIONAL COMPLETENESS FOR $\underline{B}^3$

It is an abstract of the paper which will appear in Studia Logica, vol. 33.

The study of the class of the functions $\underline{B}^3$ corresponding to D. A. Bochvar's three-valued logic [1] is subject of the present paper. In [2] V. I. Shestakov noticed that $\underline{B}^3$ can be embedded in the class of the functions corresponding to Łukasiewicz logic $Ł_3$. In [3], [4] the author examined normal forms for the functions belonging to $\underline{B}^3$ and the axiomatized algebra corresponding to B^3. We make use here of some results and symbolism from [3].

Following A. V. Kuznecov we consider the closure operation [ ] determined on the subsets of the set $\underline{B}^3$. Let $\underline{K} \subseteq \underline{B}^3$. Then we call $[\underline{K}]$ the closure of the set $\underline{K}$. $[K]$ comprises all superpositions ([5]) of functions belonging to $\underline{K}$. The set of functions $\underline{K}$ is called closed if $[\underline{K}] = \underline{K}$; the set of functions $\underline{K} \subseteq \underline{B}^3$ is called pre-complete in $\underline{B}^3$ provided that $[\underline{K}] \neq \underline{B}^3$ and for any function $F \in \underline{B}^3$ such that $F \neq \underline{K}$, $[\underline{K} \cup \{F\}] = \underline{B}^3$; a set $\underline{K}$ is said to be functionally complete in $\underline{B}^3$ if $[\underline{K}] = \underline{B}^3$.

Let $0, 1, 2$ be the logical values of the logic B_3/0 =falsehood/. By $\sim x_1$, $x_1 \cap x_2$, $x_1 \cup x_2$ we denote the functions called: internal negation, internal conjunction, and internal disjunction, respectively [1,3]. Their truth-tables are as follows:

$x_1 \cap x_2$	0	1	2	$\sim$
0	0	1	0	2
1	1	1	1	1
2	0	1	2	0

$x_1 \cup x_2$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

Let us consider now the function $J_\alpha x = \begin{cases} 0, & \text{when} \quad x \neq \alpha \\ 2, & \text{when} \quad x = \alpha \end{cases}$

where $\alpha \in \{0,1,2\}$. Let us introduce the following functions of [1]: $\neg x =_{df} J_0 x$, $\downarrow x =_{df} J_1 x$, $\vdash x =_{df} J_2 x$. The functions $\sim x_1$, $\vdash x_1$, $x_1 \Cap x_2$ constitute a basis for $\underline{B}^3$ [1].

Let F be any function from $\underline{B}^3$.

Then $F(x_1,\ldots,x_n) = J_{x_{i_1}} \Cup \ldots \Cup J_{x_{i_k}} \Cup \vdash F(x_1,\ldots,x_n)$ where $0 \leqslant i_k \leqslant n$, and $J_{x_j} =_{df} x_j \Cap \sim x_j$ (see [3]).

The functions F such that $F = F$, i.e., $k = 0$, are called external, and the functions such that $k = n$ are called properly internal, the functions such that $0 < k < n$ being called non-properly internal (cf. [1], [3]). Obviously, $\underline{B}^3 = \underline{B}^3_{ex}\ \underline{B}^3_{in}$, where $\underline{B}^3_{ex}$ is the set of external functions, and $\underline{B}^3_{in}$ is the set of all internal functions from $\underline{B}^3$.

The set of all properly internal functions from $\underline{B}^3$ is denoted by $\underline{B}^3_{in,p}$, and the set of all non-properly internal functions from $\underline{B}^3$ is denoted by the symbol $B^3_{in,imp}$.

Let us notice that $\vdash F$ may be presented in the following way: $\vdash F = G_F \Cup H_F$ where G_F is the part of J-perfect normal form of F not containing the occurrences of $\downarrow x_j$, and H_F is the part of J-perfect normal form of F every conjunctive member of which contains one occurrence of $\downarrow x_j$ at most.

We denote the set of all functions $F \in B^3_{ex}$ such that $G_F = 0$ and $H_F \neq 0$ by $\underline{B}^3_H$.

By G^*_F we denote a function homomorphic to the function G_F.

The following 11 sets of functions are pre-complete in $\underline{B}^3$.

1-5. Let $\underline{B}^3_x$, where x is T_0, T_2, S, L, or M be the set of functions $F \in \underline{B}^3$ such that G^*_F belongs to T_0, T_2, S, L or M, respectively, (where T_0, T_2, S, L, M are pre-complete sets of two-valued functions [6] preserving constant 0, constant 2, self-dual, linear, and monotonic).

6. $\underline{B}^3_{in} = \{F :$ there exists x_i such that
$F(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n) = 1\}$

7. Let us introduce the following notation:

X_F = the set of all those variables x_j of the function $F \in B^3_{in}$ such that $F(x_1,\ldots,x_{j-1},1,x_{j+1},\ldots,x_n) = 1$.

Let us take into consideration now those functions $F \in \underline{B}^3_{in,imp}$ such that $G_F = V_F \Cap K_F$, where $V^*_F = 2$, and K_F such that $K^*_F \neq 0$ and K^*_F and

$K^*_F \neq 2$ and $X_F \cap X_{V_F} = \emptyset$ and $X_{K_F} \subseteq X_F$. $X_{K_F} \subset X_F$

Let us put:
$\underline{B}^3_Q = \{F' : F' \in \underline{B}^3_{in,imp}$ and $G^*_F = const\}$
$\underline{B}^3_H = \{H : H \in B^3_{ex}$ and $G^*_H = const\}$
$\underline{B}^3_\Theta = \{F : F \in B^3_{in,imp}$ and $G_F = V_F \dot{\cap} K_F$ and $K^*_F \neq 2$ and $X_F \cap X_{V_F} = \emptyset$ and $X_{K_F} \subseteq X_F\}$.

The set $\underline{B}^3_\gamma = [\underline{B}^3_{in,p} \cup \underline{B}^3_Q \cup \underline{B}^3_H \cup \underline{B}^3_\Theta]$ is pre-complete in $\underline{B}^3$.

8. Let x^*_j and $-x^*_j$, $j = 1, 2, \ldots$ be, respectively, the variable and its negation in the sense of two-valued logic. Then $\underline{B}^3_F = [\underline{B}^3_{ex} \cup \{F : F \in \underline{B}^3_{in}$ and $G^*_F = const\} \cup \{F'$: there exists x_j such that $F' = J_{x_j} \cup \vdash F'$ and $G^*_{F'} \in \{x^8_j, -x^*_j\}\}]$ is a pre-complete set in $\underline{B}^3$.

9. $\underline{B}^3_{\dot{\cap}} = [\underline{B}^3_{ex} \cup (x_1 \dot{\cap} x_2) \cup \{F : F \in \underline{B}^3_{in}$ and $G^*_F = 0\}]$ is pre-complete in $\underline{B}^3$.

10. $\underline{B}^3_{\dot{\cup}} = [\underline{B}^3_{ex} \cup (x_1 \dot{\cap} x_2) \cup \{F : F \in \underline{B}^3_{in}$ and $G^*_F = 2\}]$ is pre-complete in $\underline{B}^3$.

11. $\underline{B}_{in,imp,1}$ is the set of all those functions F which are non-properly internal and which are of the form $F = J_{x_j} \dot{\cup} \vdash F$. The set $\underline{B}^3_C = [\underline{B}^3_{ex} \cup \underline{B}^3_{in,imp,1}]$ is pre-complete in $\underline{B}^3$.

THEOREM. A set $\underline{K} \subseteq \underline{B}^3$ is functionally complete in $\underline{B}^3$ iff $\underline{K}$ is not contained in any of the sets: $\underline{B}_{T_0}, \underline{B}_{T_2}, \underline{B}^3_S, \underline{B}^3_L, \underline{B}^3_M, \underline{B}^3_{in}, \underline{B}^3_\gamma, \underline{B}^3_F, \underline{B}^3, \underline{B}^3_C$.

REMARK 1. In [6] V.I. Shestakov examined various normal forms of the functions belonging to $\underline{B}^3_{ex}$. It can be proved that: the set of the functions $N \subseteq \underline{B}^3_{ex}$ is functionally complete in $\underline{B}^3_{ex}$ iff N is not contained in the following seven pre-complete sets: $\underline{B}^3_{ex,T_0}, \underline{B}^3_{ex,T_2}, \underline{B}^3_{ex,S}, \underline{B}^3_{ex,L}, \underline{B}^3_{ex,M}, \underline{B}^3_{ex,\neg}, \underline{B}^3_{ex,-}$, where $\underline{B}^3_{ex,\neg} = [\{\neg x_1 \dot{\cap} \neg x_2\}]$ and $\underline{B}^3_{ex,-} = [\{\bar{x}_1 \dot{\cap} \bar{x}_2\}]$.

REMARK 2. In [7] S. Halldén considered the three-valued logic C aiming at a systematic study of "nonsense". Defined connectives of the logic C are the functions: $\sim\downarrow x_1$, $\sim x_1$, $x_1 \dot{\cap} x_2$. It is easy to see that $\underline{B}^3_C \subset \underline{B}^3_\gamma$, where $\underline{B}^3_C = [\{\sim\downarrow x_1, \sim x_1, x_1 \dot{\cap} x_2\}]$.

References

[1] D. A. Bochvar, *Ob odnom trehznačnom iscislénii i égo priménénii k analizu paradoksov klassicéskogo rassirénnogo funkcional'nogo isciclénié,*

Matématicéskij sbornik, 1938, t. 4, vypusk 2.

[2] V. I. Sestakov, *O vzaimootnosenii nékotorych tréhznacnyh logicéskih iscislénij*, **Uséhi Matématicéskih nauk**, 1964, t. 19, vypusk 2 (16).

[3] V. K. Finn, *Ob aksiomatizacii nékotorych tréhznacnyh logik*, **Naucno-technicéskaá informaciá**, Seriá 2, No. 11, 1971.

[4] V. K. Finn, **Aksiomatizaciá nékotoryh tréhznacnyh iscislénij vyskazyvanij i ih algébr**, 1972, rukopis' déponirovana v VINTI AN, SSSR, 4471-71 Dep.

[5] S. V. Yablonskij, *Funkcional'nyé postroéniá v k-znacnoj logiké*, **Trudy matématicéskogo instituta in. V. A. Stéklova**, 1958, t. 51.

[6] V. J. Sestakov, *Ob odnom fragménté iscisléniá D. A. Bocvara*, **Informacionnyé voprosy semiotyki, lingvistiki i avtomaticéskogo pérévoda**, VINITI, M., 1971, vypusk 1.

[7] S. Halldén, **The logic of nonsense**, Uppsala, 1949.

All-Union Institute
of Science and Technical Information,
Moscow