Bulletin of the **S**ection of **L**ogic
Volume 2/1 (1973), pp. 12–15

reedition 2013 [original edition, pp. 12–17]

Jerzy Kotas

THE AXIOMATIZATION OF S. JAŚKOWSKI'S DISCUSSIVE SYSTEM

S. Jaśkowski in [2] and [3] has determined through interpretation a new logical system D_2, very interesting in many respects, which he has called a discussive system. L. Dubikajtis and N. C. A. da Costa in [1] gave an infinite axiom set for that system. The present paper aims at demonstrating that D_2 is a finitely axiomatizable. We shall make use of the Łukasiewicz bracketless notation. The symbols K, A, C, N, M and L will denote conjunction, disjunction, material implication, negation, possibility, and necessity, respectively. Logical systems will be treated as the sets of formulae.

In determining the system D_2 Jaśkowski employes the modal system $S5$ of Lewis. He enriches the set of logical connectives of the system $S5$ with three additional connectives K_d, C_d, and E_d which he calls discussive conjunction, discussive implication, and discussive equivalence, respectively. These logical connectives he defines as follows:

Definition 1. $K_dpq =_{df} KpMq$;

Definition 2. $C_dpq =_{df} CMpq$;

Definition 3. $E_dpq =_{df} K_dC_dpqC_dqp$.

The system D_2 is the least set of formulae α fulfilling the following conditions:

1^0. There are present in α only the signs of propositional variables and the signs K_d, C_d, E_d, A and N;

2^0. The expression $M\alpha_L$, where α_L is the expression obtained from α by eliminating the symbols K_d, C_d, E_d, according to their definitions

1-3, is a thesis of the system $S5$.

Let the symbols $LS5$ and $M - S5$ denote, respectively, the set of all those theses of the system $S5$ at the beginning of which there is the symbol L, and the set of all formulae of the system $S5$ which after being preceded by the sign M become theses of the system $S5$. Since we treat the logical systems as sets of formulae, $LS5$ and $M - S5$ are certain logical systems. Making use of C, N and L it is possible to define in a well known way all remaining logical connectives of the system $S5$; thus considering the systems $S5$, $LS5$ and $M-S5$ we can confine ourselves to the set of formulae where, apart from variables, there are only the signs C, N and L. For the same reason we may put that K_d, C_d, A, N are the only signs of the logical connectives occurring in the formulae of the system D_2.

Let A be the set consisting of the following formulae:

(A_1) $LCpCNpq$,
(A_2) $LCCpqCCqrCpr$,
(A_3) $LCCNppp$,
(A_4) $LCLpp$,
(A_5) $LCLCpqCLpLq$,
(A_6) $LCNLpLNLp$.

In further considerations we shall employ the following rules of deduction:

(R_1) : substitution rule;
(R_2) : if α, then $L\alpha$;
(R_3) : if α, then $L\alpha$;
(R_4) : if $L\alpha$, then α;
(R_5) : if $NLN\alpha$, then α.

The symbol $Cn(A; R_1, \ldots, R_n)$ denote the set of all formulae which are the consequences of the set A with respect to the rules of deduction $(R_1), \ldots,$ (R_n).

Lemma 1. $Cn(A; R_1, R_2, R_3) = LS5$.

Lemma 2. $Cn(A; R_1, R_2, R_3, R_4) = S5$.

Lemma 3. $Cn(A; R_1, R_2, R_3, R_4, R_5) = M - S5$.

Since from $C\alpha\beta \in M - S5$ and $C\beta\alpha \in M - S5$ it does not follow that $CN\alpha N\beta \in M - S5$, then the material implication C is not in the system $M - S5$ the implication as meant in [4]. For the same reasons C_d is not an implication as meant above, in the system D_2. It is easy to prove that the strict implication, denoted here by the symbol C_S, as well as the logical connective I which is defined in the following way:

Definition 4. $Ipq =_{df} NK_dANrrNANpq$ are implications in the system $M - S5$ and D_2, respectively, in the sense as determined in [4].

The interpretation i_1 of the system D_2 in the system $M - S5$, and the interpretation i_2 of the system $M - S5$ in the system D_2 we determine in the following way:

I. For any formulae α and β of the system D_2:
   i) $i_1(\alpha) = \alpha$, when α is a propositional variable,
   ii) $i_1(N\alpha) = Ni_1(\alpha)i_1(\beta)$,
   iii) $i_1(A\alpha\beta) = CNi_1(\alpha)i_1(\beta)$,
   iv) $i_1(K_d\alpha\beta) = NANi_1(\alpha)LNi_1(\beta)$,
   v) $i_1(C_d\alpha\beta) = CNLNi_1(\alpha)i_1(\beta)$;

II. For any formulae α and β of the system $M - S5$:
   i) $i_2(\alpha) = \alpha$ when α is a propositional variable,
   ii) $i_2(N\alpha) = Ni_2(\alpha)$,
   iii) $i_2(C\alpha\beta) = ANi_2(\alpha)i_2(\beta)$,
   iv) $i_2(L\alpha) = NK_dANppNi_2(\alpha)$.

Lemma 4. *$\ulcorner C_Si_1(Ipq)C_Spq\urcorner \in M - S5$ and $\ulcorner C_SC_Spqi_1(Ipq)\urcorner \in M - S5$.*

Lemma 5. *$\ulcorner Ii_2(C_Spq)Ipq\urcorner \in D_2$ and $\ulcorner IIpqi_2(C_Spq)\urcorner \in D_2$.*

From the Lemmas 4 and 5 it follows that the interpretation i_1 turns the implication I in the strict implication C_S, and the interpretation i_2 turns the implication C_S in I.

Lemma 6. *The interpretations i_1 and i_2 establish the equivalence of the systems D_2 and $M - S5$.*

Because D_2 and $M - S5$ are equivalent systems, from the notes at the end of [4], from Theorem 4 put in [5], p. 361, and from Lemmas 3 and 6 it follows

THEOREM. *D_2 is a finitely axiomatizable system.*

In the above mentioned Theorem 4 of [5] a method is given which effectively enables to obtain the axioms of a logical system if the axioms of the equivalent logical system are known. Making use of that method we obtain the following

COROLLARY. *The formulae $i_2(A_i)$, $i = 1, \ldots, 6$, $Ii_2i_1(Fpq)Fpq$, $IFpqi_2i_1(Fpq)$, where instead of the symbol F, symbols K_d, C_d, A should be put in turn, and the rules $(R_i, i_2($, $i = 1, 2, 3, 4, 5$, connected through the interpretation i_2 with the rules (R_i), $i = 1, 2, 3, 4, 5$, constitute the complete axiom set of the system D_2.*

References

[1] N. C. A. da Costa, L. Dubikajtis, *Sur la logique discursive de Jaśkowski*, **Bull. Acad. Pol. Sci.** 16 (1968), pp. 551–557.

[2] S. Jaśkowski, *Propositional calculus for contradictory deductive systems*, **Studia Soc. Scien. Toruniensis** 1 (1948), pp. 57–77.

[3] S. Jaśkowski, *On discussive conjunction in the propositional calculus for contradictory deductive systems*, **Studia Soc. Scien. Toruniensis** 1 (1949), pp. 171–172.

[4] J. Kotas, *Logical systems with implications*, **Studia Logica** 28 (1971), pp. 101–117.

[5] J. Kotas, A. Pieczkowski, *Allgemeine logische und matematische Theorien*, **Zeit. Math. Logik Grundlagen Math.** 16 (1970), pp. 353–376.

Institute of Mathematics
Nicholas Copernicus University
Toruń