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A PREDICATE CALCULUS WITH FORMULAS WHICH LOSE SENSE AND THE CORRESPONDING PROPOSITIONAL CALCULUS

It is Professor J. Ślupecki who initiated the investigations in the field of logical calculus with formulas which lose sense. The first positive solution regarding the construction of the system and its connection with the classical propositional calculus is published in [6]. Further results are presented in [5], [4], [7] and [3]. The formulation given in the article is original.

1. The predicate calculus RP

The auxiliary system in defining the tautology of the calculus W is the predicate calculus RP .

To the language of RP belong:

- (a) nominal variables x_1, x_2, \dots ,
- (b) predicate constants $P_1^1, P_2^1, \dots, P_1^2, P_2^2, \dots, P_i^m$ (the upper index informs that the predicate is m -ary),
- (c) the functors of the propositional calculus $\rightarrow, \wedge, \sim$.

By a model we mean an ordered triple

$$\underline{M} = \langle V, \{R_i^m\}, \{\bar{R}_i^m\} \rangle, \text{ where:}$$

- (a) V is a non-empty set,
- (b) $R_1^m, R_2^m, \dots, \bar{R}_1^m, \bar{R}_2^m, \dots$ are m -ary relations fulfilling the conditions:
 $R_i^m \subseteq V^m, \bar{R}_i^m \subseteq V^m, R_i^m \cap \bar{R}_i^m = \emptyset, i, m = 1, 2, \dots$,
- (c) $\{R_i^m\}, \{\bar{R}_i^m\}$ are sequences of all R_i^m and all \bar{R}_i^m ($i, m = 1, 2, \dots$).

We shall define the notion of validity.

Let φ, ψ denote any formulas of the system RP , and let v_1, v_2, \dots be any sequence of elements of v . The formulas $\varphi_{\{v_i\}} = 1$, $\varphi_{\{v_i\}} = 0$, $\varphi_{\{v_i\}} = \frac{1}{2}$, are read:

- (i) the sequence $\{v_i\}$ satisfies (in model \underline{M}) the formula φ ;
- (ii) the sequence $\{v_i\}$ falsifies (in model \underline{M}) the formula φ ;
- (iii) the sequence $\{v_i\}$ does not satisfy the formula φ , neither it falsifies that.

DEFINITION 1. The inductive definition of the formulas

$$\varphi_{\{v_i\}} = 1 \text{ and } \varphi_{\{v_i\}} = 0 :$$

- (a) $P_i^m(x_{i_1}, \dots, x_{i_m})_{\{v_i\}} = 1$ iff $\langle v_{i_1}, \dots, v_{i_m} \rangle \in R_i^m$
 $P_i^m(x_{i_1}, \dots, x_{i_m})_{\{v_i\}} = 0$ iff $\langle v_{i_1}, \dots, v_{i_m} \rangle \in \overline{R}_i^m$
 $P_i^m(x_{i_1}, \dots, x_{i_m})_{\{v_i\}} = \frac{1}{2}$ iff $\langle v_{i_1}, \dots, v_{i_m} \rangle \notin R_i^m \cup \overline{R}_i^m$
- (b) $\ulcorner \varphi \rightarrow \psi \urcorner = 1$ iff $\varphi_{\{v_i\}} = 0$ or $\varphi_{\{v_i\}} = \frac{1}{2}$ or $\varphi_{\{v_i\}} = 1$
 $\ulcorner \varphi \rightarrow \psi \urcorner = 0$ iff $\varphi_{\{v_i\}} = 1$ and $(\psi_{\{v_i\}} = 0 \text{ or } \psi_{\{v_i\}} = \frac{1}{2})$
hence it follows that $\ulcorner \varphi \rightarrow \psi \urcorner_{\{v_i\}} = \frac{1}{2}$ never holds.
- (c) $\ulcorner \varphi \wedge \psi \urcorner_{\{v_i\}} = 1$ iff $\varphi_{\{v_i\}} = 1$ and $\psi_{\{v_i\}} = 1$
 $\ulcorner \varphi \wedge \psi \urcorner_{\{v_i\}} = 0$ iff $\varphi_{\{v_i\}} = 0$ and $\psi_{\{v_i\}} = 0$ or $\varphi_{\{v_i\}} = 0$ and $\psi_{\{v_i\}} = 1$
or $\varphi_{\{v_i\}} = 1$ and $\psi_{\{v_i\}} = 0$
hence $\ulcorner \varphi \wedge \psi \urcorner_{\{v_i\}} = \frac{1}{2}$ iff $\varphi_{\{v_i\}} = \frac{1}{2}$ or $\psi_{\{v_i\}} = \frac{1}{2}$
- (d) $\ulcorner \sim \varphi \urcorner_{\{v_i\}} = 1$ iff $\varphi_{\{v_i\}} = 0$; $\ulcorner \sim \varphi \urcorner_{\{v_i\}} = 0$ iff $\varphi_{\{v_i\}} = 1$; $\ulcorner \sim \varphi \urcorner_{\{v_i\}} = \frac{1}{2}$ iff $\varphi_{\{v_i\}} = \frac{1}{2}$.

From the assumption that $R_i^m \cup \overline{R}_i^m = V$ which is not accepted here, it would follow that the given notion of validity is identical with the usual one, at the same time the formula $\varphi_{\{v_i\}} = 0$ would be the negation of the formula $\varphi_{\{v_i\}} = 1$.

DEFINITION 2. φ is valid in a model $\underline{M}(\varphi_{\underline{M}} = 1)$ iff for every sequence $\{v_i\}$, $\varphi_{\{v_i\}} = 1$.

DEFINITION 3. φ is a tautology of $RP(\varphi_{RP} = 1)$ iff for every \underline{M} , $\varphi_{\underline{M}} = 1$.

2. Propositional calculus W in which losing sense formulas can be substituted for variables

Consider the sentential language which will be called the language of W , formed by means of sentential variables $p_1^1, \dots, p_i^m, \dots$ and connectives \Rightarrow, \cdot, \neg ,

DEFINITION 4. $\varphi = \tau(\varphi^*)$ iff φ and φ^* are formulas of RP and W respectively and φ is the result of replacing $\Rightarrow, \cdot, \neg, p_i^m$ in φ^* by $\rightarrow, \wedge, \sim, P_i^m(x_{i_1}, \dots, x_{i_m})$ respectively.

DEFINITION 5. φ is a tautology of the system $W(\varphi_W^* = 1)$ iff $\varphi_{RP} = 1$.

The set of all tautologies of the system W will be denoted by τ_W . We call the set of all tautologies understood this way, the system W . The matrix \underline{M}_W is of the form

$$\underline{M}_W = \langle \{0, 1, \frac{1}{2}\}, \{1\}, \Rightarrow, \cdot, \neg \rangle$$

where the symbols " \Rightarrow ", " \cdot ", " \neg " denote the operations determined by means of the following tables:

| \Rightarrow | 1 | 0 | $\frac{1}{2}$ | \cdot | 1 | 0 | $\frac{1}{2}$ | \neg | |
|---------------|---|---|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 1 | 1 | 0 | 0 | 1 | 1 | 0 | $\frac{1}{2}$ | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | $\frac{1}{2}$ | 0 | 1 |
| $\frac{1}{2}$ | 1 | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

(In Bochvar's calculus B_3 there are operations determined by analogous tables, cf. [1], [2].)

Let us consider the following set of formulas:

- I. B. Sobociński's axioms of two-valued propositional calculus with implication and conjunction as the only connectives axioms for implication and negation.
- II. Axioms for implication and negation
 - A1. $p \Rightarrow (-p \Rightarrow q)$
 - A2. $p \Rightarrow -(-p)$
 - A3. $-(-p) \Rightarrow p$
 - A4. $p \Rightarrow [(q \Rightarrow -q) \Rightarrow -(p \Rightarrow q)]$.
- III. The laws of negating conjunctions

- A5. $-(p \cdot q) \Rightarrow -(q \cdot p)$
- A6. $-(p \cdot q) \Rightarrow [(-p \Rightarrow p) \Rightarrow p]$
- A7. $p \cdot -q \Rightarrow -(p \cdot q)$
- A8. $-p \cdot -q \Rightarrow -(p \cdot q)$

IV. The rejected axiom

$$\overline{A1.} \quad -(p \cdot -p).$$

As acceptance rules we regard: modus ponens and substitution; as rejection rules: Łukasiewicz's rejection – by – detachment rule and rejection – by – substitution rule.

The notations:

T_W – the set of all consequences of accepted axioms

T_W^{-1} – the set of all rejected propositions

F_W – the set of all formulas of the system W .

The following theorems are true:

THEOREM 1. $T_W \subseteq \tau_W$

THEOREM 2. $T_W \subseteq E(\underline{M}_W)$

THEOREM 3. $T_W^{-1} \subseteq F_W = \tau_W$

THEOREM 4. $T_W^{-1} \subseteq F_W - E(\underline{M}_W)$

THEOREM 5. $T_W \cup T_W^{-1} = F_W$

THEOREM 6. $T_W \cap T_W^{-1} = \emptyset$

THEOREM 7. $\tau_W \subseteq T_W$

THEOREM 8. $E(\underline{M}_W) \subseteq T_W$

The basic theorem of the system W is an immediate conclusion from Theorems 1, 7, 2 and 8:

THEOREM 9. $T_W = \tau_W = E(\underline{M}_W)$

The matrix \underline{M}_W is also strongly adequate for the logic W , cf. [3].

Further theorems select classes of formulas which are theses of the calculus W from among theses of the classical propositional calculus.

THEOREM 10. *Every thesis of the classical propositional calculus in which only implication and conjunction functors occur is a thesis of the calculus W .*

THEOREM 11. *Let in formulas φ and ψ only the functors of conjunction and negation occur, and let both φ and ψ be built up by means of the same variables. Then if the equivalence $(\varphi \Rightarrow \psi) \cdot (\psi \Rightarrow \varphi)$ is a thesis of the classical propositional calculus, then it is also a thesis of the system W .*

To the fact that in the calculus W not all laws of extensionality are maintained testifies that the formula

$$(p \Rightarrow q) \Rightarrow (-p \Rightarrow -q)$$

is not a thesis of that calculus. What is true is the following theorem which informs in what sense the extensionality rule holds in the system W .

THEOREM 12. $\ulcorner (p \Leftrightarrow q) \cdot (-p \Leftrightarrow -q) \Rightarrow [\varphi \Leftrightarrow \varphi(p//q)] \urcorner \in T_W$.

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