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THE SEQUENT GENTZEN SYSTEM FOR m-VALUED LOGIC

In paper [6] we defined the rules of decomposition of formulas in m-valued propositional calculus and predicate calculus, which contain the functors $\vee, \wedge, \rightarrow, \neg, D_1, \ldots, D_{m-1}, E_0, \ldots, E_{m-1}$, and correspond algebraically to Post algebras. That system of rules is analogous to Gentzen NK-calculus for classical two-valued logic [1]. The aim of the present paper is to formulate a system of rules of inference analogous to Gentzen LK-calculus.

Being the most useful for logic, the definition of Post algebra as formulated by G. Rousseau [5] and applied by other authors [2], [3], [4], [6] will be used. In their papers the concept of set of formulas of this system, as well as the concepts of realization and valuation, are defined in a natural way. As in [6] we will consider only the realization of the language L_m for m-valued logic into the m-element Post algebra P_m.

We will apply the following definitions and lemmas formulated in [6]:

A formula a is called e_k-valid, if, for every valuation
$v : V \rightarrow P_m, va \geqslant e_k$.

A formula a is called e_k-unvalid, if, for every $v, va < e_k$, i.e.
$va \leqslant e_{k-1}$

Lemma 1. *Formula a is e_k-valid iff $D_k(a)$ is a tautology.*

Lemma 2. *Formula a is e_k-unvalid iff formula $D_k(a)$ is refutable, (i.e. iff $\neg D_k(a)$ is a tautology).*

In [6] rules of inference have the form

$$\frac{\Gamma^1, \ldots, \Gamma^i}{\Gamma}$$

where $\Gamma, \Gamma^1, \ldots, \Gamma^i$ are finite sequence of formulas, Γ is called the conclusion and $\Gamma^1, \ldots, \Gamma^i$ – the premises of the rule. Now, we shall define rules for pairs of sequences, constituting the conclusion and premises of the rule, i.e., rules of the form:

$$\frac{\Gamma^1 \vdash \Delta^1; \ldots; \Gamma^1 \vdash \Delta^i}{\Gamma \vdash \Delta}$$

The (possibly empty) pair of sequences $\Gamma - \Delta$ will be further called a sequent with the antecedent Γ and the consequent Δ. A sequent $S \vdash T$ is called a tautology (e_k-valid, e_k-unvalid, refutable), if $\bigwedge_{s_i \in S} \to \bigvee_{t_i \in T} t_i$ is a tautology (e_k-valid, e_k-unvalid, refutable).

We obviously assume that $\bigwedge_{s_i \in S} s_i = V$ if $S \neq \emptyset$, and $\bigvee_{t_i \in T} t_i$ if $T = \emptyset$.

The rules in present form corresponding to the rules in [6]. will have the form:

$(A\vee)$ $\dfrac{\Gamma, D_i(a) \vdash \Delta; \Gamma, D_i(b) \vdash \Delta}{\Gamma, D_i(a \vee b) \vdash \Delta}$

$(S\vee)$ $\dfrac{\Gamma \vdash \Delta, D_i(a), D_i(b)}{\Gamma \vdash \Delta, D_i(a \vee b) \vdash \Delta}$

$(S\wedge)$ $\dfrac{\Gamma, D_i(a), D_i(b) \vdash \Delta}{\Gamma, D_i(a \wedge b) \vdash \Delta}$

$(S\wedge)$ $\dfrac{\Gamma \vdash \Delta, D_i(a); \Gamma \vdash \Delta, D_i(b)}{\Gamma \vdash \Delta, D_i(a \wedge b)}$

$(A \to)$ $\dfrac{\Gamma \vdash \Delta, D_1(a); \Gamma, D_1(b) \vdash \Delta, D_2(a); \ldots; \Gamma, D_{i-1}(b) \vdash \Delta, D_i(a) \Gamma, D_i(b) \vdash \Delta}{\Gamma, D_i(a \to b) \vdash \Delta}$

$(S \to)$ $\dfrac{\Gamma, D_1(a) \vdash \Delta, D_1(b); \ldots; \Gamma, D_i(a) \vdash \Delta, D_i(b)}{\Gamma \vdash \Delta, D_i(a \to b)}$

$(A\neg)$ $\dfrac{\Gamma \vdash \Delta, D_i(a)}{\Gamma, \neg D_i(a) \vdash \Delta}$ $(S\neg)$ $\dfrac{\Gamma, D_i(a) \vdash \Delta}{\Gamma \vdash \Delta, D_i(a)}$

(Ai) $\dfrac{\Gamma, D_j(a) \vdash \Delta}{\Gamma, D_j(D_j(a)) \vdash \Delta}$ (Si) $\dfrac{\Gamma \vdash \Delta, D_j(a)}{\Gamma \vdash \Delta, D_i(D_j(a))}$

(Aij) $\dfrac{D_1(a) \vdash D_1(a)}{D_i(E_j) \vdash}$ for $i>j$ (Sij) $\dfrac{D_1(a) \vdash D_1(a)}{\vdash D_i(E_j)}$ for $i<j$

where $\underline{a}$ is a fixed propositional variable or predicate.

For predicate calculus we assume moreover the following rules with the natural restriction for variables:

$(AE)\ \dfrac{\Gamma, D_i(a(x)) \vdash \Delta}{\Gamma, D_i(V_\xi a(\xi)) \vdash \Delta}$ $(SE)\ \dfrac{\Gamma \vdash \Delta, D_i(a(x))}{\Gamma \vdash \Delta, D_i(V_\xi a(\xi))}$

$(AU)\ \dfrac{\Gamma, D_i(a(x)) \vdash \Delta}{\Gamma, D_i(\Delta_\xi a(\xi)) \vdash \Delta}$ $(SU)\ \dfrac{\Gamma \vdash \Delta, D_i(a(x))}{\Gamma \vdash \Delta, D_i(\Delta_\xi a(\xi))}$

For each of these rules the conclusion is a tautology if and only if all premises are tautologies. This is quite obvious, because each of the above rules is an exact transformation of some rule in [6].

We add the following structural rules:

$(AA)\ \dfrac{\Gamma \vdash \Delta}{\Gamma, a \vdash \Delta}$ $(SA)\ \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, a}$

$(AR)\ \dfrac{\Gamma, a, a \vdash \Delta}{\Gamma, a \vdash \Delta}$ $(SR)\ \dfrac{\Gamma \vdash \Delta, a, a}{\Gamma \vdash \Delta, a}$

$(AM)\ \dfrac{\Gamma', a, b, \Gamma'' \vdash \Delta}{\Gamma', b, a, \Gamma'' \vdash \Delta}$ $\dfrac{\Gamma' \vdash \Delta', a, b, \Delta''}{\Gamma \vdash \Delta', b, a, \Delta''}$

It is obvious that for every structural rule if the premise is a tautology, the conclusion is a tautology.

We assume the following definitions:

An indecomposable formula is a formula of the form $D_i(p)$, where p is a propositional variable in propositional calculus or an atom formula in predicate calculus and $1 \leqslant i \leqslant m-1$.

An axiom is a sequent of the form $D_j(p) \vdash D_i(p)$, where $1 \leqslant i \leqslant j \leqslant m-1$ and p has the same meaning as above.

An indecomposable sequent is a sequent in which all formulas are indecomposable.

A fundamental sequent is a sequent in which all formulas are indecomposable.

A fundamental sequent is a sequent obtained from an axiom by means of structural rules.

A proof of the sequent $S \vdash T$ is a sequence of sequents $X_1, \ldots, X_n$, where $X_n = S \vdash T$ and every X_j for $1 \leqslant j \leqslant n$ is an axiom or is obtained from some $X_{i_1}, \ldots, X_{i_l}$ $(i_1, \ldots, i_l < j)$ by means of a rule of inference given above.

If $S = (s_1, \ldots, s_n)$, let $D_i(S) = D_i(s_1), \ldots, D_i(s_n)$.

Thus, we can formulate the following property:

LEMMA 1. *The sequent $S \vdash T$ is e_k-valid iff $D_j(S) \vdash D_j(S) \vdash D_j(T)$ is tautology for every $1 \leqslant j \leqslant k$.*

PROOF. The sequent $S \vdash T$ is e_k-valid if and only if for every valuation v, $v(\bigwedge_{s_i \in S} s_i \to \bigvee_{t_j \in T} t_j) \leqslant e_k$, i.e. iff $vD_k(\wedge s_i \to V t_j) = V$. In every Post algebra

$$D_k(a \to b) = \bigwedge_{j=1}^{k} (D_j(a) \to D_j(b))$$

i.e. the validity of all implications of the form $D_j(S) \to D_j(T)$ for $j \leqslant k$ is equivalent to the validity of the formula $D_k(S \to T)$.

LEMMA 2. *The sequence $S \vdash T$ is e_k-unvalid iff*

1. *S is an e_1-valid formula,*
2. *T is an e_k-unvalid formula,*
3. *for every i $(1 \leqslant i \leqslant k-1) : D_j(T) \to D_{i+1}(S)$ is a valid formula.*

PROOF. This property follows form Lemma 1 and the property (p_{14}) in [6] valid in every Post algebra"

$$\neg D_i(a \to b) = D_1(a) \cap \bigwedge_{j=1}^{i-1} (D_{j+1}(a) \cup \neg D_j(b)) \cap \neg D_i(b)$$

It is worth noticing that the condition 1 and 2 have a quite natural meaning.

From [6] and [1] it is quite obvious that every sequent consisting of Boolean formulas in propositional or predicate calculus is a tautology if and only if it has a proof. By combining this fact with Lemmas 3 and 4 we obtain the following theorems:

A sequent $S \vdash T$ is e_k-valid iff for every $i \leqslant k$ the sequent $D_i(S) \vdash D_i(T)$ has a proof. A sequent $S \vdash T$ is e_k-unvalid iff the sequents $\vdash D_1(S)$, $D_k(T) \vdash D_i(T) \vdash D_{i+1}(S)$ for $1 \leqslant i \leqslant k-1$ have proofs.

EXAMPLE 1. In the m-valid propositional calculus or m-valued predicate calculus the sequent $E_1 \vdash E_0$ is obviously even e_1-unvalid, but $D_i(E_1) \vdash D_i(E_0)$ is a tautology for $i \geqslant 2$ and only the sequence $D_1(E_1) \vdash D_1(E_0)$ is refutable.

For $i \leqslant 2$	$\underline{D_1(\underline{a}) \vdash D_1(\underline{a})}$
from $(Ai1)$	$\underline{D_i(E_1 \vdash}$
from (SA)	$D_i(E_1) \vdash D_i(E_0)$

A sequent $S \vdash T$ consisting of Boolean formulas only is refutable if and only if for every valuation $v : v \bigwedge_{s_i \in S} s_i = V$ and $v \bigvee_{t_j \in T} t_j = \bigwedge$.

EXAMPLE 2. The sequent $\vdash x_0 \rightarrow x_1, \ldots, x_k \rightarrow x_{k+1}$, where x_i are propositional variables is e_k-valid but not e_{k+1}-valid.

In this example we will use the concept of a diagram of a sequent defined in a natural way. It is obvious that every sequent of formulas in propositional calculus has a finite diagram obtained without the use of the rules (AA) and (SA) in which final sequents are indecomposable. In the decomposition of the considered formula only one rule $(S \rightarrow)$ is used.

The sequent $D_i(x_0 \rightarrow x_1), \ldots, D_i(x_k \rightarrow x_{k+1})$ decomposes into the sequents: $D_1(x_k) \vdash D_i(x_0 \rightarrow x_1), \ldots, D_i(x_{k-1} \rightarrow x_k), D_1(x_{k+1}), \ldots, D_i(x_k) \vdash D_i(x_0 \rightarrow x_1), \ldots, D_i(x_{k-1} \rightarrow x_k), D_i(x_{k+1})$. By repeating the rule $(S \rightarrow)$ and the structural rule (SM) we obtain as final in the diagram the sequents $D_{j_1}(x_k), \ldots, D_{j_i}(x_0) \vdash D_{j_1}(x_{k+1}), \ldots, D_{j_1}(x_1)$ for $1 \leqslant j_p \leqslant i$. If $i = k+1$, there exist a sequence of indices $\{j_p\}_{p=1}^{k+1}$ for which the sequent is not fundamental, namely the sequence $j_1 = 1, \ldots, j_{k+1} = k+1$. If $i \leqslant k$, such sequence does not exist and every indecomposable sequent in the diagram is fundamental.

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