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AN ABSOLUTE FIRST ORDER PREDICATE CALCULUS

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A language with two kinds of individual variables – free $x, v, \ldots$, and bound $x, y, \ldots$ – is built. What is called formula is quasi-formula without free occurrences of bound variables in it. Substitution is denoted by $F^w_x A$, all substitutions are correct, i.e. such that no one occurrence of w in A is within the range of the binding quantifier x. As a sequent is meant the expression of the form $\Gamma \to \Delta$, where Γ and Δ are the sequence of formula which may be empty. If Δ consists of one formula or if Δ is empty, the sequent $\Gamma \to \Delta$ will be called singular, otherwise it will be multiple.

Now we shall give the classification of the logical rules of deduction (with singular sequents), dividing them into 4 groups.

I		II	
IIR	$\dfrac{A, \Gamma \to B}{\Gamma \to A \supset B}$	RIR	$\dfrac{\Gamma \to A \supset B}{A, \Gamma \to B}$
ICR	$\dfrac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \,\&\, B}$	RCR	$\dfrac{\Gamma \to A \,\&\, B}{\Gamma \to A \text{ and } \Gamma \to B}$
IDL	$\dfrac{A, \Gamma \to \Theta \quad B, \Gamma \to \Theta}{A \vee B, \Gamma \to \Theta}$	RDL	$\dfrac{A \vee B, \Gamma \to \Theta}{A, \Gamma \to \Theta \text{ and } B, \Gamma \to \Theta}$
INR	$\dfrac{A, \Gamma \to}{\Gamma \to \neg A}$	RNR	$\dfrac{\Gamma \to \neg A}{A, \Gamma \to}$

$I\forall R$ $\dfrac{\Gamma \to A}{\Gamma \to \forall_x F_x^w A}$ $R\forall R$ $\dfrac{\Gamma \to \forall_x F_x^w A}{\Gamma \to F_t^w A}$

$I\exists L$ $\dfrac{A, \Gamma - \to \Theta}{\exists_x F_x^w A, \Gamma \to \Theta}$ $R\exists L$ $\dfrac{\exists_x F_x^w A, \Gamma \to \Theta}{F_t^w A, \Gamma \to \Theta}$

III IV

IIL $\dfrac{\Gamma \to \ B, \Delta \to \Theta}{A \supset B, \Gamma, \Delta \to \Theta}$ RIL $\dfrac{A \supset B, \Gamma \to A}{\Gamma \to}$

ICL $\dfrac{A, \Gamma \to \Theta \text{ or } B, \Gamma \to \Theta}{A \ \& \ B, \Gamma \to \Theta}$ RCL $\dfrac{A \ \& \ B, \Gamma \to \Theta}{A, B, \Gamma \to \Theta}$

IDR $\dfrac{\Gamma \to A \text{ or } \Gamma \to B}{\Gamma \to A \vee B}$ RDR $\dfrac{\Gamma \to A \vee B \ A, \Delta \to}{\Gamma, \Delta \to B}$

INL $\dfrac{\Gamma \to A}{\neg A, \Gamma \to}$ RNL $\dfrac{\neg A, \Gamma \to}{\Gamma \to A}$

$I\forall L$ $\dfrac{F_t^w A, \Gamma \to Theta}{\forall_x F_x^w A, \Gamma \to \Theta}$

$I\exists R$ $\dfrac{\Gamma \to F_t^w A}{\Gamma \to \exists_x F_x^w A}$

(The abbreviation IIR stand for: the introduction of the implication from the right, the remaining abbreviations are of analogous meaning).
In the case of $I\forall R$ and of $I\exists L$ it is required that the variable w should not occur in Θ and Γ. The sequent $A \to A$ is called the basic one. Structural rules of the proof are: permutation on the left side PL shortening on the left sL, and the cutting C which is oh the form:

$$\frac{\Gamma \to M \ \ \Delta_1, M, \Delta_2 \to \Theta}{\Delta_1, \Gamma, \Delta_2 \to \Theta}$$

The proof is in the form of a tree.

The absolute logical predicate calculus of sequents SLA is define in terms of the rules of groups I and III, and of the structural rules. The rules of the group I and II together with the structural rules determine the absolute calculus of sequents SNA. Those systems are equivalent.

The absolute system is at the bottom of the classification of logical systems: minimal, intuitionistic, classical. There are two possible approaches:

enriching SLA with the rules of group IV or enriching SLA with additional structural rules.

In particular, enriching SLA with the rule RCL gives the minimal system SLM, and adding the rule RDR to SLM gives the intuitionistic SLJ, while the addition of the rule RIL to SLJ gives a classical system. The system SLA enriched with the rule RNL will be called a strong system SLS.

In another approach we obtain a minimal system adding the thinning rule from the left TL to SLA, we get an intuitionistic system by adding the thinning rule from the right TR to the minimal one. In order to carry the classification further on two multiple sequents should be taken into consideration. Let us notice that the strong system is Gentzenian calculus LK with multiple sequents without thinning rules.

For SLA the cut elimination theorem takes place.

That theorem cannot be proved by means of ordinary methods, i.e., the lemma of "cut" exchange by "mix" is not true. The idea of the proof consists in the exchange of "cuts" by generalized "mix" $\dfrac{\Gamma \to M \ \Delta_1, M, \Delta_2 \to \Theta}{\Delta_{1_M}, \Gamma, \Delta_{2_M} \to \Theta}$ where Δ_{1M} and Δ_{2M} are results of plotting some (not necessarily all – as in an ordinary case) occurrences of the formula M in Δ_1 and in Δ_2. The cut elimination theorem also holds for SLS (and, as it is well known, for the minimal, intuitionistic, and classical systems).

Let us now construct a Hilbert's type absolute system with the notion of the proof of HA.

1. $A \supset A$
2. $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$
3. $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$
4. $(A \supset (A \supset C)) \supset (A \supset C)$
5. $(C \supset A)\ \&\ (C \supset B) \supset (C \supset A\ \&\ B)$
6. $A\ \&\ B \supset A$
7. $A\ \&\ B \supset B$
8. $(A \supset C)\ \&\ (B \supset C) \supset (A \vee B \supset C)$
9. $A \supset A \vee B$
10. $B \supset A \vee B$
11. $(A \supset \neg B) \supset (B \supset \neg A)$
12. $\forall_x F_x^w A \supset F_t^w A$
13. $F_t^w A \supset \exists_x F_x^w A$

14. $\forall_x F_x^w (C \supset A) \supset (C \supset \forall_x F_x^w A)$
15. $\forall_x F_x^w (A \supset C) \supset (\exists_x F_x^w A \supset C)$

Proof rules:

$$\frac{A \; A \supset B}{B}, \qquad \frac{A}{\forall_x F_x^w A}, \qquad \frac{AB}{A \,\&\, B}$$

By a string of the assumptions we shall understand a sequence of assumptions (accurate to the permutation of its members). The notion of the proof is different from the ordinary one and it uses the induction with respect to the height of proof and the number of assumptions.

1. If E is an axiom, then E is the deduction from the empty string of assumption.
2. E is a deduction from the string of assumptions E.
3. If α is a deduction from the string of assumptions Γ, A – the last formula of α, β is the deduction from the string of assumptions Δ, and $A \supset B$ is the last formula of β, then $\frac{\alpha\beta}{B}$ is the deduction from the string of assumptions Γ, Δ.
4. If α is a deduction from the string of assumptions Γ, a variable w does not occur in the formulas of Γ and A is the last formula of α, then $\frac{\alpha}{\forall_x F_x^w A}$ is the deduction from the string of assumptions Γ.
5. If α and β are the deductions from the empty strings of assumptions, A and B are their last formulas, then $\frac{\alpha\beta}{A \,\&\, B}$ is the deduction from the empty string of assumptions.

In HA the following deduction theorem holds: for every deduction α from the string Γ, A with the last formula B there exists a deduction of A B from the string Γ_A, where Γ_A results from plotting some occurrences of A in Γ.

In order to obtain a minimal system from HA the axiom scheme should be attached to it: $(A \supset B) \supset (C \supset (A \supset))$. Adding $\neg\neg A \supset A$ to HA gives the strong system HS; adding the scheme $A \supset (\neg A \supset B)$ to the minimal system gives the intuitionistic system, and adding the Pierce's law to that system gives the classical system. Let us notice that enrichting the strong system with Pierce's law gives the classical system, too.

The absolute system in the form of natural deduction system NA is being constructed in the following way:

First the notion of the forest of formulae is introduced:

1. If A is the formula, then A is the forest of formulae;
2. If α and β are forests of formulae, A and E are formulae, then

$$\frac{\alpha}{E}, \ \frac{\alpha\ \beta}{E}, \ \frac{|\alpha}{E}, \ \frac{|\alpha}{E}, \ \frac{A\ |\alpha\ |\beta}{E}$$

are the forests of formulae.

Let us now determine the notion of deduction for NA.

1. A is a deduction from the string of assumptions A;
2. If α is a deduction of A from a string of assumptions Γ, and E follows directly from A, then $\frac{\alpha}{E}$ is a deduction from a string of assumptions Γ;
3. If α is a deduction from a string of assumptions Γ, β is a deduction from a string of assumptions Δ, and E follows directly from A, B – the last formulas of α and *beta* respectively, then $\frac{\alpha\ \beta}{E}$ is a deduction from the string of assumptions Γ, Δ;
4. If α is a deduction from a string of assumptions Γ, and A is the last formula of α, and a variable w does not occur in the formulas of Γ, then $\frac{\alpha}{\forall_x F_x^w A}$ is a deduction from Γ.
5. If α is a deduction from a string Γ, and β is a deduction from Γ then $\frac{\alpha, \beta}{A\ \&\ B}$ is a deduction from Γ.
6. If α is a deduction of B from a string of assumptions A, Γ, then $\frac{|\alpha}{A \supset \beta}$ is a deduction a string of assumptions Γ_A.
7. If α is a deduction of E from a string of assumptions A, Γ, and β is a deduction of E from a string B, Γ, then $\frac{A \vee B|\alpha\ |\beta}{E}$ is a deduction from a string of assumptions $A \vee B$, Γ.
8. If α is a deduction of C from a string A, Γ, and w does not occur in the formulas of Γ and C, then $\frac{\exists_x F_x^w A|\alpha}{E}$ is a deduction from a string of assumptions $\exists_x F_x^w A$, Γ.

There hold theorems of SLA, SNA, NA and HA equivalences. Finally we shall formulate two unsolved problems:

1. Is the problem of dedidability for the propositional part of the absolute system solvable?
   If we enrich SLA with the rule $\frac{\Gamma \to A \ \Delta \to A}{\Gamma \ \Delta \to A}$, then we shall get a system for which cut elimination theorem holds and for the propositional part of which the problem of decidability is solved.
2. Does definability theorem hold for the absolute and strong systems? If the answer is yes, it results a modified Beth's theorem.

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