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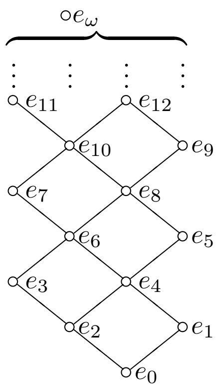
REMARKS ON INTERMEDIATE LOGICS WITH AXIOMS CONTAINING ONLY ONE VARIABLE

This is a summary of the result reported in December 1972 at the seminar of the Department of Logic of Jagiellonian University held by Professor S. J. Surma in Cracow. The full text with detailed proofs will appear in the forthcoming number of Zeszyty Naukowe Uniwersytetu Jagiellońskiego, seria Prace z Logiki.

Nishimura presents in [6] the free pseudo-Boolean algebra with one-element free generating set as the subalgebra of the Lindenbaum-algebra of the intuitionistic propositional logic (INT) consisting of the equivalence classes of the formulas φ_n, $n = 0, 1, \ldots, \omega$ defined as follows: $\varphi_0 = p \wedge \neg p$, $\varphi_1 = \neg p$, $\varphi_2 = p$, $\varphi_3 = \neg\neg p$, $\varphi_4 = p \vee \neg p$, $\varphi_{2n+5} = \varphi_{2n+3} \Rightarrow \varphi_{2n+2}$, $\varphi_{2n+6} = \varphi_{2n+3} \vee \varphi_{2n+1}$, $\varphi_\omega = p \Rightarrow p$ (p is a propositional variable). Therefore the free pseudo-Boolean algebra with one-element free generating set can be visualized by means of the following diagram (the point e_n of the diagram corresponds to the equivalence class of the formula φ_n, $n = 0, 1, \ldots, \omega$).



Anderson proved in [1] that for every $n = 0, 1, \ldots,$ such that $n \neq 2$, the intermediate logic which results by adding φ_{2n+9} to the axioms of INT has the disjunction property (i.e. the disjunction of any two unprovable formulas is also unprovable). The aim of this paper is to show that it holds also for $n = 2$. We introduce some generalization of Troelstra's concept of sum of pseudo-Boolean algebra (see [7]) which enables us to prove this fact in the uniform way for all the cases $n = 1, 2, \ldots$. Let us begin with some notations, conventions and definitions. The German capitals $\mathcal{A}, \mathcal{L}, \ldots$ will be used to denote pseudo-Boolean algebras (called in the sequel-algebras). The corresponding Latin capitals $A, B, \ldots$ denote the domains of $\mathcal{A}, \mathcal{L}, \ldots$. The letter $\mathcal{R}$ is reserved for the two-element Boolean algebra. The symbols $\mathbf{1}_{\mathcal{A}}$, $\mathbf{0}_{\mathcal{A}}$, $\leqslant_{\mathcal{A}}$ denote the unit element, the zero element and the lattice ordering in $\mathcal{A}$. If $\mathcal{A}$ is an algebra, $X \subseteq A$ then by $(X]$ and $[X)$ we denote the smallest ideal and the smallest filter in $\mathcal{A}$ containing X. We will write $(a]$ and $[a)$ instead of $(\{a\}]$ and $[\{a\})$ respectively. If $\mathcal{A}$ and $\mathcal{L}$ are algebras, $a \in A$ and $b \in B$ then we say that $\mathcal{A}$ is $[a, b]$ – associable to $\mathcal{L}$ iff the ordered sets $\langle [a), \leqslant_{\mathcal{A}} \rangle$ and $\langle (b], \leqslant_{\mathcal{L}} \rangle$ are isomorphic. If $\mathcal{A}$ is $[a, b]$ – associable to $\mathcal{L}$ the one can require that $A \cap B = [a) = (b]$ (if not then one can use appropriate isomorphic copies) and in such the case the ordered set $\langle A \cup B, \leqslant_{\mathcal{A}} \cup \leqslant_{\mathcal{L}} \cup (\leqslant_{\mathcal{A}} \odot \leqslant_{\mathcal{L}}) \rangle$ ($\odot$ denotes superposition) is a relatively pseudocomplemented lattice with the zero element. The pseudo-Boolean algebra corresponding to such a lattice will be denoted by $\mathcal{A} \oplus \mathcal{L}$, $\mathcal{A}\oplus$ instead of $\mathcal{A} \oplus \mathcal{L}[\mathbf{1}_{\mathcal{A}}, \mathbf{0}_{\mathcal{L}}], \mathcal{A} \oplus \mathcal{R}$ respectively. The definition of $\mathcal{A} \oplus \mathcal{L}$ is to be found in Troelstra's [7] and the definition of $\mathcal{A}\oplus$ already in Jaśkowski's [4]. Let us denote the free pseudo-Boolean algebra with one-element free generating set by $\mathcal{F}$ and the quotient algebra $\mathcal{F}/[e_n)$ by $\mathcal{F}_n$, $n = 0, 1, \ldots$.

Lemma 0. *$\mathcal{F}_{2n+1}$ is embeddable into $\mathcal{A}$ iff there exists a valuation v in $\mathcal{A}$ such that $v(\varphi_{2n+1}) = \mathbf{1}_{\mathcal{A}}$ and $v(\varphi_{2n}) \neq \mathbf{1}_{\mathcal{A}}$.*

Corollary 0. *The following conditions are equivalent:*

(i) $\varphi_{2n+3} \in E(\mathcal{A})$ *($E(\mathcal{A})$ denotes the content of $\mathcal{A}$),*

(ii) $E(\mathcal{A}) \not\subseteq E(\mathcal{F}_{2n+1})$,

(iii) There is no subalgebra of some quotient algebra of $\mathcal{A}$ which is isomorphic to $\mathcal{F}_{2n+1}$.

Corollary 1. *The following conditions are equivalent:*

(i) $\varphi_{2n+2} \in E(\mathcal{A})$,

(ii) $E(\mathcal{A}) \not\subseteq E(\mathcal{F}_{2n+1})$ *and* $E(\mathcal{A}) \not\subseteq E(\mathcal{F}_{2n+3})$,

(iii) There is no subalgebra of some quotient algebra of $\mathcal{A}$ which is isomorphic to $\mathcal{F}_{2n+1}$ or to $\mathcal{F}_{2n+3}$.

Let us observe that for arbitrary $\mathcal{A}$, the product $\mathcal{A} \times \mathcal{R}$ is $[\langle \mathbf{1}_{\mathcal{A}}, \mathbf{0}_{\mathcal{R}} \rangle, e_2]$-associable to $\mathcal{F}$. Therefore starting from the algebra $\overline{\mathcal{A}} = (\mathcal{A} \times \mathcal{R}) \oplus \mathcal{F}[\langle \mathbf{1}_{\mathcal{A}}, \mathbf{0}_{\mathcal{R}} \rangle, e_2]$ we define the algebras $\mathcal{A}_n$, $n = 0, 1, \ldots$ as follows: $\mathcal{A}_0 = \overline{\mathcal{A}}/[\mathbf{0}_{\overline{\mathcal{A}}})$, $\mathcal{A}_1 = \mathcal{A}/[\langle \mathbf{0}_{\mathcal{A}}, \mathbf{1}_{\mathcal{R}} \rangle)$, $\mathcal{A}_{n+2} = \overline{\mathcal{A}}/[e_n)$.

Lemma 1.

(i) $\mathcal{R}_n$ *is isomorphic to* $\mathcal{F}_n$,

(ii) If $\mathcal{A}$ *is non-degenerate then the element* $\langle \mathbf{1}_{\mathcal{A}}, \mathbf{0}_{\mathcal{R}} \rangle$ *generates in* $\overline{\mathcal{A}}$ *a subalgebra isomorphic to* $\mathcal{F}$,

(iii) $E(\mathcal{A}_{2n+6}) = E(\mathcal{A}_{2n+3}) \cap E(\mathcal{A}_{2n+1})$.

Lemma 2. *If* $\mathcal{F}_{n+8}$ *is embeddable into* $\mathcal{A} \times \mathcal{R}$ *then it is also embeddable into* $\mathcal{A}$.

Lemma 3.

(i) If $m \geqslant 6$ *and* $\mathcal{L}$ *is a subalgebra of* $\mathcal{A}_m$ *such that* $|B| \geqslant 3$ *and* $[e_1][e_m] \notin B$ *then* $\mathcal{L}$ *has a form* $\mathcal{L} \oplus \vartheta$ *where both* $\mathcal{L}$ *and* ϑ *are non-degenerate,*

(ii) If $m \geqslant 6$ *and* $\mathcal{A}$ *is non-degenerate then* $\mathcal{F}_{2n+8}$ *is embeddable into* $\mathcal{A}_m$ *iff* $m = 2n + 8$.

From Lemma 0, 1, 2, 3 (ii) we get the following:

Corollary 2. *If* $\varphi_{2n+11} \in E(\mathcal{A}\oplus)$ *then* $\varphi_{2n+11}\ E(\mathcal{A}_{2n+7})$.

If α is a formula then by $\varphi_n(\alpha)$ we denote the formula resulting by substitution of α for p in φ_n.

Lemma 4. *If there exist a valuation* v_0 *in* $\mathcal{R}$ *such that* $v_0(\alpha) = \mathbf{0}_{\mathcal{R}}$ *and a valuation* v_1 *in* $\mathcal{A}\oplus$ *such that* $v_1(\alpha) = \mathbf{1}_n$ *then there exists a valuation* v *in* $\overline{\mathcal{A}}$ *such that* $v(\alpha) = \langle \mathbf{1}_{\mathcal{A}}, \mathbf{0}_{\mathcal{R}} \rangle$.

Let the symbol FR denotes the set of all formulas. If $X \subseteq FR$ and $\alpha \in FR$ then we write $X \vdash \alpha$ to express that α is derivable from the formulas from $INT \cup X$ by means of the detachment rule. By $Sb(X)$ we denote the set of all substitution instances of the formulas from X. We will write $Sb(\alpha)$ instead of $Sb(\{\alpha\})$. By Corollary 2 and Lemma 4 we get the following:

Theorem 0. $Sb(\varphi_{2n+11} \vdash \varphi_{2n+8}(\alpha)$ *iff* $Sb(\varphi_{2n+11}) \vdash \neg\neg\alpha \Rightarrow \alpha$.

Following Kleene [5] we define the relation $X|\alpha$ ($X \subseteq FR$ – and $\alpha \in FR$) by the following conditions:

(i) $X|p$ iff $X \vdash p$ for every propositional variable p,

(ii) $X| \neg\neg\alpha$ iff not $X \Vdash \alpha$ or $X \vdash \neg\neg\alpha$ ($X \Vdash \alpha$ means that $X|\alpha$ and

$X \vdash \alpha$),

(iii) $X|\alpha \Rightarrow \beta$ iff not $X \Vdash \alpha$ or $X|\beta$,

(iv) $X|\alpha \mathbin{\wedge\!\!\!\wedge} \beta$ iff $X|\alpha$ and $X|\beta$,

(v) $X|\alpha \mathbin{\vee\!\!\!\vee} \beta$ iff $X \Vdash \alpha$ or $X \Vdash \beta$ (for origins of this definition see Harrop [2] and [3]).

If $X \subseteq FR$ then by $Cn(X)$, $Pr(X)$ and $Rd(X)$ we will mean $\{\alpha : X \vdash \alpha\}$, $\{\alpha : X|\alpha\}$ and $\{\alpha : X \Vdash \alpha\}$ respectively.

Lemma 5.

(i) $INT \subseteq \cap(Pr(X) : X \subseteq FR)$,

(ii) $Pr(Cn(X)) = Pr(X)$,

(iii) $Rd(Cn(X)) = Rd(X) = Cn(Rd(X))$,

(iv) $X \subseteq Pr(X)$ *iff* $Rd(X) = Cn(X)$.

Theorem 1. *The following conditions are equivalent:*

(i) $Cn(X)$ *has the disjunction property,*

(ii) $Cn(X) = Rd(X)$,

(iii) $X \subseteq Pr(X)$,

(iv) $Cn(X) \subseteq Pr(X)$.

The condition (iii) of Theorem 1 is a very useful tool for proving the disjunction property for intermediate logics.

Lemma 6. $X|\varphi_{2n+6}(\alpha)$ *iff* $X \vdash \varphi_{2n+3}(\alpha)$ *or* $X \vdash \varphi_{2n+1}(\alpha)$.

Theorem 2. $Cn(Sb(\varphi_{2n+11}))$ *has the disjunction property for every* $n = 0, 1, \ldots,$

Proof. Form Theorem 0 and Lemma 6 it follows that $Sb(\varphi_{2n+11}) \subseteq Pr(Sb(\varphi_{2n+11}))$ thus the desired result follows by Theorem 1. Q.E.D.

Remark. It is easy to prove that $=_| \alpha$, $=_| \; =_| \alpha \notin E(\mathcal{R})$ iff $=_| \; =_| \alpha \vee \vee =_| \alpha \notin E((\mathcal{R} \times \mathcal{R})\oplus)$. This gives that $Sb(\varphi_9) \vdash \varphi_6(\alpha)$ iff $Sb(\varphi_9) \vdash \varphi_3(\alpha)$ or $Sb(\varphi_9) \vdash \varphi_1(\alpha)$. Therefore the fact that also $Cn(Sb(\varphi_9))$ has the disjunction property follows analogously by Lemma 6 and Theorem 1. Thus one can easily infer that if every formula belonging to a set $X \subseteq FR$ contains only one variable then $Cn(Sb(X))$ has the disjunction property iff either $Cn(Sb(X)) = Cn(Sb(\varphi_{2n+9}))$ for some $n = 0, 1, \ldots$ or $Cn(Sb(X)) = INT$ or $Cn(Sb(X)) = FR$.

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