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ON THE DEGREE OF COMPLETENESS OF POSITIVE LOGIC

We shall use the terminology and notion of our paper "Remarks on intermediate logics..." (this volume, pp. 58–64). A formula $\alpha\ FR$ is called positive iff the negation sign does not occur in α (allowed are only the connectives $\Rightarrow$, $\wedge\!\!\!\wedge$ and $\vee\!\!\!\vee$). By PFR we denote the set of all positive formulas and by POS the well-known positive logic (i.e. $INT \cap PFR$, see [5]). The symbol Cp denotes the consequence operation determined by POS, the substitution rule and the detachment rule. We say that $X \subseteq PFR$ is a positive intermediate logic iff $Cp(X) \subseteq X \neq PFR$. It was proved by Jankov [2] that there exists a sequence $\{\alpha_n : n = 1, \ldots\} \subseteq PFR$ such that $\alpha_n \Rightarrow \alpha_m \notin POS$ iff $m \neq n$. In the direction of strengthening this result we will show that there exists a sequence $\{\pi_n : n = 0, 1, \ldots\} \subseteq PFR$ such that $\pi_n \notin Cp(\{\pi_m : m \neq n\})$, $n = 0, 1, \ldots$ The fact above allows us to prove that the degree of completeness of POS is $2^{\aleph_0}$ (i.e. there exist $2^{\aleph_0}$ positive intermediate logics) and construct some not finitely approximable, not finitely axiomatizable and undecidable positive intermediate logics (for intermediate logics analogous results where obtained by Jankov [1]. If $\mathcal{A}$ is a pseudo-Boolean algebra then by $Ep(\mathcal{A})$ we denote $E(\mathcal{A}) \cap PFR$. It is well-known that every positive intermediate logic is identical with $Ep(\mathcal{A})$ for some non-degenerate pseudo-Boolean algebra $\mathcal{A}$ and conversely if a pseudo-Boolean algebra $\mathcal{A}$ is non-degenerate then $Ep(\mathcal{A})$ is a positive intermediate logic. By positive embedding (p-embedding) of $\mathcal{A}$ into $\mathcal{L}$ we mean an embedding of the positive reduct of $\mathcal{A}$ (i.e. reduct obtained by dropping the pseudocomplement operation $\neg_{\mathcal{A}}$) into the positive reduct of $\mathcal{L}$. The terms:
p-subalgebra, p-generating set will be used in the analogous way. Let $\mathcal{A}$ be finite and strongly compact (i.e. such that in $A - \{\mathbf{1}_{\mathcal{A}}\}$ there exists the

greatest element $\star_{\mathcal{A}}$). Following Jankov [3] for the algebra $\mathcal{A}$ we define the positively characteristic formula $p\chi(\mathcal{A})$ such that:

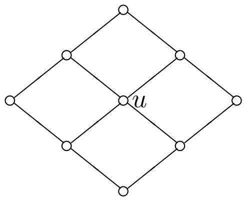
$$\begin{aligned} p\chi(\mathcal{A}) = \quad & (\bigwedge ((a_x \Rightarrow a_y)) && \Leftrightarrow && a_x \to_{\mathcal{A}} y : x, y \in A) \\ & \bigwedge ((a_x \bigwedge a_y) && \Leftrightarrow && a_x \wedge_{\mathcal{A}} y : x, y \in A) \\ & \bigwedge ((a_x \bigvee a_y) && \Leftrightarrow && a_x \vee_{\mathcal{A}} y : x, y \in A)) \Rightarrow a_{\star_{\mathcal{A}}} \end{aligned}$$

LEMMA 0. (see [3]) *Let $\mathcal{A}$ be a finite and strongly compact algebra, let $\mathcal{L}$ be an algebra. Then the following conditions are equivalent:*

(i) $\mathcal{A}$ *is p-embeddable into a quotient algebra of* $\mathcal{L}$.

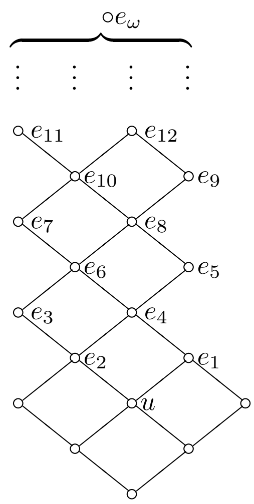
(ii) $Ep(\mathcal{L}) \subseteq Ep(\mathcal{A})$,

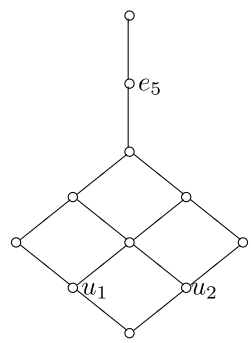
(iii) $p\chi(\mathcal{A}) \notin Ep(\mathcal{L})$.

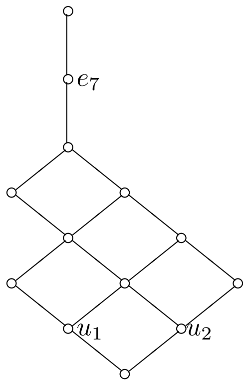


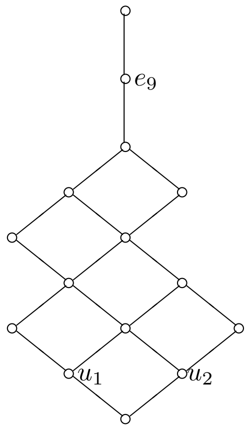
It is easy to observe that the algebra $\mathcal{F}_3 \times \mathcal{F}_3$ (see diagram) is $[u, e_4]$ – associable to the algebra $\mathcal{F}$. We denote $(\mathcal{F}_3 \times \mathcal{F}_3) \oplus \mathcal{F}[u, e_4]$ by $\mathcal{R}$ and $\mathcal{R}/[e_{2n+5}))\oplus$ by $\mathcal{R}_n$, $n = 0, 1, \ldots$.

Let us visualize the algebras $\mathcal{R}$, $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_2$ by diagrams.









LEMMA 1. *If* $n \neq m$ *then* $\mathcal{R}_n$ *cannot be p-embedded into a quotient algebra of* $\mathcal{R}_m$.

Let us denote the positively characteristic formula $p\chi(\mathcal{R}_n)$ by π_n, $n = 0, 1, \ldots$. By Lemma 0 and Lemma 1 we get the following:

THEOREM 0. $\pi_n \notin Cp(\{\pi_n : m \neq n\})$.

PROOF. Applying Lemma 0 to the statement of Lemma 1 we get that $\{\pi_m : m \neq n\} \subseteq E_p(\mathcal{R}_n)$. Since by Lemma 0 it follows that $\pi_n \notin E_p(\mathcal{R}_n)$ then the proof is finished. Q.E.D.

THEOREM 1.

(i) The degree of completeness of POS is $2^{\aleph_0}$,

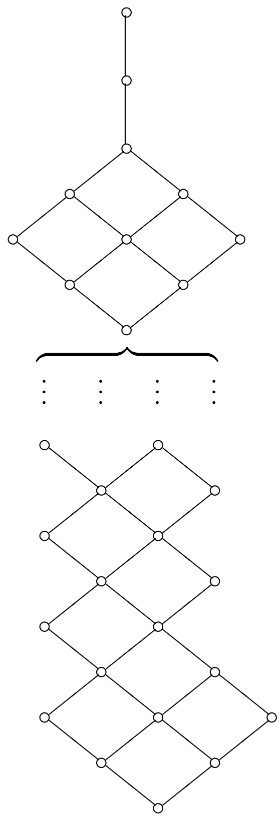
(ii) Some positive intermediate logics are not finitely axiomatizable,

(iii) For every degree of unsolvability there exists a positive intermediate logic of higher or equal degree.

PROOF. Let us denote $Cp(\{\pi_n : n \in I\})$ by $\pi(I)$, $I \subseteq \{0, 1, \ldots\}$. By Theorem 0 it follows that $\pi(I)$ is finitely axiomatizable iff I is finite and also that $|\{\pi(I) : I \subseteq \{0, 1, \ldots\}\}| = 2^{\aleph_0}$. Since $\{\pi_n : n = 0, 1, \ldots\}$ is a recursive subset of PFR then observing that $\pi_n \in \pi(I)$ iff $n \in I$ we know that the degree of unsovability of $\pi(I)$ is higher or equal to that of I. Q.E.D.

REMARK. $\mathcal{R}_n$ is p-generated by $\{u_1, u_2, e_{2n+5}\}$ (see diagram) and therefore one can find a positive formula $\overline{\pi}_n$ with three propositional variables such that $Cp(\overline{\pi}_n) = Cp(\pi_n)$.

Let α and β be positive formulas defined as follows: $\alpha = b \vee (b \Rightarrow (c \vee (c \Rightarrow \vee(d_i \Rightarrow d_j : i,j = 1,2,3,\ i \neq j))))$, $\beta = \alpha \vee \pi_0$ (we require that α and π_0 have no common propositional variable). We will show that no finite algebra separates β from $Ep(\mathcal{R} \oplus \mathcal{R}_0)$ and therefore $Ep(\mathcal{R} \oplus \mathcal{R}_0)$ is not finitely approximable since $\beta \notin Ep(\mathcal{R} \oplus \mathcal{R}_0)$. Let us visualize the algebra $\mathcal{R} \oplus \mathcal{R}_0$ by means of the diagram.



LEMMA 2.

(i) $\beta \notin Ep(\mathcal{R} \oplus \mathcal{R}_0)$,
(ii) $\mathcal{R}_0$ cannot be p-embedded into $\mathcal{R} \oplus \mathcal{R}_0/\Phi$ if the filter Φ is non-trivial,
(iii) If v is a refuting valuation of β in $\mathcal{R} \oplus \mathcal{R}_0$ then the image of the set of propositional variables of β under v p-generates $\mathcal{R} \oplus \mathcal{R}_0$.

THEOREM 2. *$Ep(\mathcal{R} \oplus \mathcal{R}_0)$ is not finitely approximable.*

PROOF. Suppose that $\mathcal{A}$ is a finite algebra such that $Ep(\mathcal{R} \oplus \mathcal{R}_0) \subseteq Ep(\mathcal{A})$ and $\beta \notin Ep(\mathcal{A})$. It is easy to see that one can find a strongly compact algebra satisfying such conditions. Therefore by Lemma 0 we get that $\mathcal{A}$ is p-embeddable into $\mathcal{R} \oplus \mathcal{R}_0/\Psi$ for some filter Ψ. Hence $\pi_0 \notin Ep(\mathcal{R} \oplus \mathcal{R}_0/\Psi)$ and applying Lemma 0 we obtain that $\mathcal{R}_0$ is p-embeddable into $\mathcal{R} \oplus \mathcal{R}_0/\Phi$ for some filter Φ such that $\Psi \subseteq \Phi$. By Lemma 2(ii) we know that the filter Φ must be trivial which implies that $\mathcal{A}$ is p-embeddable into $\mathcal{R} \oplus \mathcal{R}_0$. Thus β can be refuted in a finite p-subalgebra of $\mathcal{R} \oplus \mathcal{R}_0$ which contradicts Lemma 2(iii). Q.E.D.

PROBLEM. Kuznecov, Gerciu [4] used an algebra very similar to $\mathcal{R} \oplus \mathcal{R}_0$ in order to obtain a finitely axiomatizable and not finitely approximable intermediate logic. Is $Ep(\mathcal{R} \oplus \mathcal{R}_0)$ finitely axiomatizable?

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