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AXIOMATIZABILITY OF FINITE MATRICES (on the basis of Wajsberg's work [7])

Terminological explanations. S denotes the set of all propositions built of propositional variables $p, q, r, \dots$ (possibly with indices) with the help of functors of implication C and negation N.

$M = \langle A, B, C_M, N_M\rangle$ is a logical matrix where $A \cup B$ is a domain and B a set of distinguished values. A matrix is finite when its domain is finite, and normal when it satisfies the condition: if $x \in B$, $y \in A$ then $C_M(x, y) \in A$. $E(M)$ denotes the content of matrix M, $Cn(X)$ denotes the set of propositions X. A matrix M is axiomatizable if there is a finite set of propositions X such that $E(M) = Cn(X)$. $u_M \sim v$ holds when the propositions u and v have the same value in a matrix M for every valuation. The symbol $C^k pq$ is defined inductively:

$C^0 pq = q$, $C^{k+1} pq = CpC^k pq$.

M. Wajsberg's work [4] has been published in 1935 (written in 1933). Its main result is the proof of the theorem about axiomatizability of finite matrices, stating:

> if M is a finite, normal logical matrix in which the formulas $CCpqCCqrCpr$, $CCqrCCpqCpr$, $CCqqCpp$, $CCpqCNqNp$, $CNqCCpqNp$ are satisfied, then a matrix M is axiomatizable.

This result, without the proof and with the formula $CCqrCpp$ instead of $CCqqCpp$, was cited as early as in 1930 in the work of Łukasiewicz and Tarski [ŁT]. The proof of the theorem under discussion, with some small changes in notation, constitute the content of R. Ackermann's [A].

It is perhaps worth the mention about the theorem given by Wajsberg in his work [3]:

> if the proposition $C^k pp$ is satisfied for certain k in a finite matrix M then this matrix cannot be axiomatized using the formulas containing only two different propositional variables.

Let now M be a matrix in which $\overline{\overline{A \cup B}} = m$.
Let i, j be the least integers for which
$CC^{i+j}CpqrC^{i}Cpqr \in E(M)$
Let $V_m = \{u \in S : Zm(u) \subset \{p_1, p_2, \ldots, p_m\}\}$
$Aq = \{u \in V_m :$ for each $v \in V_m$ there exists exactly one proposition $u \in Aq$ such that $v \underset{M}{\sim} u\}$
$Aqm = (Aq \cap E(M)) \cup \{CCuvw, CwCuv : u, v, w \in Aq$ and $w \underset{M}{\sim} Cuv\} \cup \{CNuv, CvNu : u, v \in Aq$ and $v \underset{M}{\sim} Nu$
$u_n = C^{i}Cp_kp_lC^{i}Cp_lp_kCqr, 1 \leqslant k < l \leqslant m+1,\ n = 1, 2, \ldots, \frac{m(m+1)}{2}$
$Ax = Aqm \cup \{CCpqCCqrCpr, CCqrCCpqCpr, CCqqCpp, CCpqCNqNp, CC^{i+j}CpqrC^{i}Cpqr, C^{j}u_1C^{j}u_2 \ldots C^{j}u_{\frac{m(m+1)}{2}-1}C^{j+1}u_{\frac{m(m+1)}{2}}C^{2i}CqqCqr$

The following theorem holds:

THEOREM. $E(M) = Cn(Ax)$.

The proof of the inclusion $Cn(Ax) \subset E(M)$ constitutes §§3.3, 3.4 and the Theorem 22 of the work [4]. The proof of the second part, i.e. that $E(M) \subset Cn(Ax)$ is carried by the induction for the number of different propositional variables occurring in the propositions from $E(M)$. Wajsberg proves first ([4], thm 14) that $E(M) \cap V_m \subset Cn(Ax)$. The inductive step ([4], thm 23) shows that assuming the theorem holds for propositions containing at most $s-1$ different propositional variables it holds for propositions containing s variables too.

References

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