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WAJSBERG ON THE FIRST-ORDER PREDICATE CALCULUS FOR THE FINITE MODELS
(on the basis of Wajsberg's work [4])

Wajsberg considers the first-order predicate calculus for the finite models in [4]. The author assumes the formalisation of the first-order predicate calculus presented in [HA] with the following changes and completions. By "formulas of the first-order predicate calculus" are understood only such formulas which do not contain individual constants, predicate constants nor variable symbols of the propositional calculus. The special role play the formulas in which appear only one-argument predicates, which will be denoted as Cl-formulas here. True formulas in a model containing exactly k elements (k is a natural number creator than 0) are called k-true formulas, while k-true formulas which are not at the same time $k + 1$-true formulas are described as exactly k-true formulas. The expression "A follows from $B, C, \ldots$" is equivalent to the expression "A is provable in the first-order predicate calculus with $B, C \ldots$ as new axioms added".

Wajsberg proves the following theorem:

THEOREM I. *From every exactly k-true formula follows every k-true formula.*

The proof goes as follows.

LEMMA 1. *There is certain exactly k-true Cl-formula, from which follows every k-true formula.*

To prove the above lemma Wajsberg constructs exactly k-true Cl-formula, which has the following schema:

(S_k) $\qquad$ $A_1 \vee \ldots \vee A_k$

where
$$A_i(1 \leqslant i \leqslant k)$$

has the following form

$$F_i(x_i) \rightarrow F_i(x_{i+1}) \vee \ldots \vee F_i(x_{k+1})$$

For example, for $k = 2$ from (S_k) we get the formula

$$[F_1(x_1) \rightarrow (F_1(x_2) \vee F_1(x_3))] \vee (F_2(x_2) \rightarrow F_2(x_3))$$

The proof of Lemma 1 consists in showing that from (S_k) follows every k-true formula. While proving it Wajsberg uses only propositional and predicate calculi. The same concerns the proofs of the next lemmas.

LEMMA 2.   *From every exactly k-true Cl-formula follows every k-true Cl-formula.*

In order to prove Lemma 2 it is necessary to consider some properties of Cl-formulas. Wajsberg uses here results due to Herbrand [H].

From each conjunction with n members

$$F_1(c) \wedge \ldots \wedge F_n(x)$$

can be formed 2^n of different conjunctions with n members by negation of any of the expressions $F_j(x)$ $(1 \leqslant j \leqslant n)$. These conjunctions are elementary n-degree conjunctions. Let K_i $(i \leqslant n)$ denote the elementary i-degree conjunction. The expressions $\bigvee_x K_i(x)$ and $-\bigvee_x K_i(x)$ are respectively positive and negative basis i-degree formula and disjunction ofdifferent basis formulas of the equal degree is the normal disjunction of the same degree.

Herbrand proved the following theorem:

Every Cl-formula without free variables, which has exactly n predicates can be reduced to the form of the conjunction of the normal n-degree disjunctions.

Owing to this theorem the proof of Lemma 2 can be reduced to the proof of the fact that every k-true normal disjunction follows from every exactly k-true normal disjunction.

The next two lemmas state some structural properties of the k-true and exactly k-true normal disjunction necessary for the proof of Lemma 2.

LEMMA 2A.

(1) If A is exactly k-true normal disjunction then A contains exactly $k + 1$ different basis formulas beginning with negation and no number of A is the negation of any other member.

(2) If A is the k-true normal disjunction, then A is the theorem, or contains at least $k + 1$ basic formulas beginning with negation.

LEMMA 2B. *If for the given formulas $A_1 \ldots A_2n$ the following expressions are the theorems*
(1) $-(A_i \wedge A_j)$ $i, j \leqslant 2^n$, $i \neq j$
(2) $A_1 \vee \ldots \vee A_2n$
and $K_1, \ldots, K_2n$ are the elementary conjunctions built up of the predicates $F_1, \ldots, F_n$, then after substitution of F_l ($l \leqslant n$) by a certain, the same for all $K_1, \ldots, K_2n$, expressions, the m-th conjunction ($m \leqslant 2^n$) will be equivalent to A_m.

From Lemma 1 and Lemma 2 follows, that
From every exactly k-true formula follows every k-true formula.
To finish the proof of Theorem 1 it is enough to point that holds:

LEMMA 3. *From every exactly k-true formula follows certain exactly k-true formula.*

In the considerations of the predicate calculus for the finite models usually the following is stated.

THEOREM II. *From every k-true formula follows every k-true formula.*

Wajsberg's theorem establishes the essential completion of Theorem II. (Com. [HB], p. 120).

References

[H] J. Herbrand, *Recherches sur la théorie de la démonstration*, **Travaux de la Sociéte des Sciences et de Lettres de Varsovie**, 1930.

[HA] D. Hilbert and W. Ackermann, **Grunzüge der theoretischen Logik**, Berlin 1928.

[HB] D. Hilbert and P. Bernays, **Grundlage der Mathematik, Band I**. Berlin 1968, Zweite Auflage.

[4] M. Wajsberg, *Untersuchungen über den Funktionenkalkül für endliche Individuenbereiche*, **Mathematische Annalen** 108 (1933), Heft 2, pp. 118–128.

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