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MV_k ALGEBRAS

This is an abstract of the paper that will be published in Studia Logica.

In this paper we introduce the notion of MV_k algebras, which are the algebraic interpretation of Łukasiewicz's logics. MV_k algebras have advantages of k-valued algebras of Łukasiewicz and MV algebras as well. We will use the notion "k-valued algebra of Łukasiewicz" in the sense of the definition 3 from [4] but for this we are going to give the following modification of that definition: the condition $M2$: $\sigma_i(x) \subset \sigma_{i+1}(x)$ we replace with $M2$: $\sigma_i(x) \supset \sigma_{i+1}(x)$.

Definition 1. We shall say that the algebra

(1) $\underline{V}_k = \langle V, +, \cdot, \cup, \cap, -, 0, 1, \sigma_1, \sigma_2, \ldots, \sigma_{k-1}\rangle$ $(k \geqslant 2)$ is an MV_k algebra provided that

$K1$. The reduct $\underline{Ł}_k = \langle V, \cup, \cap, -, 0, 1, \sigma_1, \sigma_2, \ldots, \sigma_{k-1}\rangle$ is the k-valued algebra of Łukasiewicz

$K2$. The reduct $\underline{V} = \langle V, +, \cdot, -, 0, 1\rangle$ is the MV algebra (see [2]) and

$x \cup y = x \cdot \overline{y} + y$

$x \cap y = (x + \overline{y}) \cdot y$

$K3$. $\sigma_1(x) = (k-1)x + (x\ \overline{x})$, where $(k-1)x =^{df} x + x + \ldots + x(k-1) times$

$K4$. If $x \neq \overline{y}$, then $\sigma_1(x) \cdot \sigma_1(y) \leqslant \sigma_1(x \cdot y + \overline{x} \cdot \overline{y})$

$K5$. $\sigma_i(x) \cdot \sigma_{k-i}(x) \leqslant \sigma_1(x \cdot x)$ for any $i \in \{1, 2, \ldots, k-1\}$.

Definition 2. We shall say that MV_k algebra $\underline{V}_k$ is centered provided that the k-valued algebra of Łukasiewicz $\underline{Ł}_k$ is centered i.e. there exists $(k-2)$ elements $a_2, a_3, \ldots, a_{k-1} \in V$ such that the following condition holds:

$$(2) \qquad \sigma_i(a_j) = \begin{cases} 1 & \text{for} \quad 1 \leqslant i < j \\ 0 & \text{for} \quad j \leqslant i \leqslant k-1. \end{cases}$$

If $M_k = \langle A_k, \rightarrow, \vee, \wedge, \neg, \{1\}\rangle$ is the k-valued Łukasiewicz's matrix, then the algebra

(3) $\underline{C}_k = \langle A_k, +, \cdot, \cup, \cap, \neg, 0, 1, \sigma_1, \sigma_2, \ldots, \sigma_{k-1}\rangle$

where

$$x + y = \neg x \rightarrow y, \qquad x \cup y = x \vee y$$
$$x \cdot y = \neg(x \rightarrow \neg y), \quad x \cap y = x \wedge y$$

and

$$\sigma_i(\frac{j}{k-1}) = \begin{cases} 1 & \text{for} \quad 1 \leqslant i < j+1 \\ 0 & \text{for} \quad j+1 \leqslant i \leqslant k-1 \end{cases}$$

is the centered MV_k algebra.

LEMMA 1. *Each linearly ordered MV_k algebra is isomorphic with the algebra $\underline{C}_k$.*

DEFINITION 3.

(i) $\emptyset \neq \underline{J} \subseteq V$ is called an ideal of the MV_k algebra V_k iff $\underline{J}$ is an ideal of the MV algebra $\underline{V}$ i.e. provided that
 (c1) if $x, y \in \underline{J}$, then $x + y \in \underline{J}$,
 (c2) if $x \leqslant y$ and $y \in \underline{J}$, then $x \in \underline{J}$ (see [2]).

(ii) If $\underline{J}$ is an ideal of the MV_k algebra $\underline{V}_k$, then put $x \approx_{\underline{J}} y$ $(x, y \in V)$ iff $d(x, y) \in \underline{J}$, where $d(x, y) = \overline{x} \cdot y + x \cdot \overline{y}$ ($\approx_{\underline{J}}$ is then the congruence relation on the MV algebra $\underline{V}$ – see [2]).

LEMMA 2. *For every ideal $\underline{J}$ of the MV_k algebra $\underline{V}_k$ we have $x \in \underline{J}$ iff $\sigma_i(x) \in \underline{J}$ for every $i \in \{1, 2, \ldots, k-1\}$.*

LEMMA 3. *The relation $\approx_{\underline{J}}$, where $\underline{J}$ is an ideal of the algebra $\underline{V}_k$, is the congruence relation on $\underline{V}_k$.*

DEFINITION 4. The ideal $\underline{P}$ of the MV_k algebra $\underline{V}_k$ is called first ideal of this algebra if $\underline{P}$ is the first ideal of the MV algebra i.e. (f1) $\underline{P} \neq V$, (f2) for every $x, y \in V$ either $x \cdot \overline{y} \in \underline{P}$ or $\overline{x} \cdot y \in \underline{P}$.

LEMMA 4. (cf. [3]) *If $\underline{P}$ is the first ideal of the MV algebra $\underline{V}$, then $\underline{V}/\underline{P}$ is a linearly ordered MV algebra.*

From Lemmas 4 and 1 we obtain the following:

LEMMA 5. *If $\underline{V}_k$ is an arbitrary MV_k algebra and $\underline{P}$ is the first ideal of it, then $\underline{V}_k/\underline{P}$ is isomorphic with the MV_z algebra $\underline{C}_z$ ($z \leqslant k$). If k is even, then z is even too.*

As a particular case of the last lemma we obtain the following:

LEMMA 6. *If $\underline{V}_k^c$ is a centered MV_k algebra and $\underline{P}$ is the first ideal of it, then $\underline{V}_k^c/\underline{P}$ is isomorphic with the algebra $\underline{C}_k$.*

For the case given in Lemma 5 we receive the following notation: We write $\underline{C}_z^k$ for MV_z algebra $\underline{C}_z$ ($z \leqslant k$), which is written in the same manner as MV_k algebra (that means that in this structure we find $(k-1)$ endomorphisms not necessarily different, but in the way that the conditions (ii) of Definition 3 from [4] were fulfilled – there are many such representations for the fixed algebra $\underline{C}_z$ and the quotient algebra $\underline{V}_k/\underline{P}$ from Lemma 5 is one them).

C. C. Chang has obtained the following result for MV algebras:

LEMMA 7. *Let $\underline{V}^0 = \langle V, +, \cdot, 0, 1\rangle$ be an arbitrary MV algebra. Then for every $a \in V$ there exist the prime ideal $\underline{P}_a$ such that $a \notin \underline{P}_a$.*

From the last lemma and Birkhoff's [1] it follows:

THEOREM 1. (Chang's representation theorem for MV algebras, cf. [3]) *Every MV algebra $\underline{V}^0$ is isomorphic with the subdirect product of the linearly ordered MV algebras.*

Since every first ideal of MV algebra $\underline{V}$ is the first ideal of the MV_k algebra $\underline{V}_k$ we also get Lemma 7 for MV_k algebras. Then from Lemma 5 and Lemma 7 (for MV_k algebras) and Birkhoff's [1] we get

THEOREM 2. (The representation theorem for MV_k algebras) *Every MV_k algebra $\underline{V}_k$ is isomorphic with the subdirect product MV_z algebras $\underline{C}_z^k$ ($z \leqslant k$). If k is even, then z are even too.*

The particular case of the last theorem obtained by using Lemma 6 is the following representation theorem for centered MV_k algebras:

THEOREM 3. *Every centered MV_k algebra $\underline{V}_k^c$ is isomorphic with the subdirect product MV_k algebras $\underline{C}_k$.*

The algebra of formulas
(4) $\underline{L} = \langle L, \rightarrow, \vee, \wedge, \neg \rangle$

will be called the language of the Łukasiewicz's sentential calculi. Let us now consider the algebra

(5) $\underline{U} = \langle U, \Rightarrow, \vee, \wedge, \neg \rangle$
similar to $\underline{L}$ and such that U is an (partially) ordered set containing the maximal element $\underline{1}$.

DEFINITION 5. If $X \subseteq L$ is an arbitrary set of formulas, then put
(6) $Cn_{\underline{U}}(X) = \{\alpha \in L |$ for every homomorphism $h : \underline{L} \rightarrow \underline{U}$ if $hX \subseteq \{\underline{1}\}$, then $h\alpha = \underline{1}\}$.

We say that $Cn_{\underline{U}}$ is the consequence operation determined by $\underline{U}$.

From the well known McNaughton's criterion the following lemma can easily be proved:

LEMMA 8. *In the matrix M_k $(k \geqslant 2)$ there are definable the endomorphisms $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$ be means of $\rightarrow$ and $\neg$.*

LEMMA 9. *In the MV_k algebra $\underline{C}_k$ $(k \geqslant 2)$ one can define the implication of ŁUkasiewicz by the formula*

(7) $x \rightarrow y = \overline{x} + y.$

LEMMA 10. $Cn_{\underline{C}_k} = Cn_{M_k}$ $(k \geqslant 2)$.

From the last lemma and from Theorem 3 we obtain the following:

THEOREM 4. $Cn_{\underline{V}_k^c} = Cn_{M_k}$ *for an arbitrary centered MV_k algebra $\underline{V}_k^c$* $(k \geqslant 2)$.

REMARKS.

(i) For $k = 2, 3, 4$ the notions "k-valued algebra of Łukasiewicz" and "MV_k algebra" coincide.

(ii) The analogical result to the above mentioned theorem cannot be obtained for the arbitrary (not necesserily centered) MV_k algebras.

References

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