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GENERALIZED KRIPKE MODELS

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A Kripke model for the modal logic $S4$ a triple $(A, \leqslant, \Vdash)$ such that $(A, \leqslant)$ is a partially ordered set and $\Vdash$ is a relation between elements of A and expressions of the language of $S4$. $a \Vdash \alpha$ is defined by induction on the form of the expression α, e.g. $a \Vdash \Box\alpha$ if and only if for every a' : If $a \leqslant a'$ then $a' \Vdash \alpha$. We can try to formalize the right side of this definition by $\forall x_1(a \leqslant x_1 \rightarrow x_1 \Vdash \alpha)$. Since this definition of $a \Vdash \Box\alpha$ does not depend on the form of α we can restrict our attention to the case $\alpha = p_0$. The formula $\forall x_1(a \leqslant x_1 \rightarrow x_1 \Vdash p_0)$ is almost a formula of some elementary language as usual in the model theory of classical logics. The only unusual part is $x_1 \Vdash p_0$. So there arise three simple questions.

1) What is the propositional variable p from the point of view of classical model theory?
2) What is $\Vdash$ from the point of view of classical model Theory?
3) What does $a \Vdash p$ mean?

Ad 1) Note that the meaning of p in the model $(A, \leqslant, \Vdash)$ is determined by the set $\Vdash (p) = \{a \in A : a \Vdash p\} \subseteq A$. So we can regard p as a variable for subsets of A, i.e. p is a unary relational symbol.

Ad 2) Note that $\Vdash$ is determined by the sequence of the sets $\Vdash (p_i)(i < \omega)$. So we can regard $\Vdash$ as an interpretation of the unary relational symbols p_i $(i < \omega)$ as subsets of A. Henceforth p_i $(i < \omega)$ will be regarded as a unary relational symbol and will be regarded as an interpretation of all the symbols p_i $(i < \omega)$.

Ad 3) $a \Vdash p$ if and only if $a \in\Vdash (p)$ and therefore
$a \Vdash p$ if and only if $(A, \leqslant, \Vdash, a) \models p(a)$.

Therefore $a \Vdash \Box p_0$ if and only if
$(A, \leqslant, \Vdash, a) \models \forall x_1(a \leqslant x_1 \to p_0(x_1))$

and we have obtained a definition of the relation $a \Vdash \Box p_0$ in purely model theoretic terms. So the formula

$$\forall x_1(x_0 \leqslant x_1 \to p_0(x_1))$$

is in $S4$ associated with the symbol $\Box$. In a similar way we can associate $\neg p_0(x_0)$ and $p_0(x_0) \wedge p_1(x_0)$ with $\neg$ and $\wedge$ respectively. This shows a natural way to generalize the concept of Kripke models.

Before describing this generalization we state some conventions. For every similarity type τ $L(\tau)$ is the elementary language based on the symbols of τ and on the individual variables x_i $(i < \omega)$. For every language L $F_0(L)$ denotes the set of all formulas of L which contain at most the variable x_0 free. Now let $\underline{K}$ be an arbitrary class of models of a fixed type σ. In the case of $S4$ $\underline{K}$ is the class of all partially ordered sets. Let $\underline{functor}$ be a nonempty subset of $F_0(L(\sigma \cup \{p_i : i < \omega\}))$. In the case of $S4$ $\underline{functor} = \{H_\neg, H_\wedge, H_\vee, H_\to, H_\leftrightarrow, H_\Box, H_\Diamond\}$ where $H_\neg = \neg p_0(x_0)$, $H_\wedge = p_0(x_0) \wedge p_1(x_0)$, $H_\vee = p_0(x_0) \vee p_1(x_0)$, $H_\to = p_0(x_0) \to p_1(x_0)$, $H_\leftrightarrow = p_0(x_0) \leftrightarrow p_1(x_0)$, $H_\Box = \forall x_1(x_0 \leqslant x_1 \to p_0(x_1))$ and $H_\Diamond = \exists x_1(x_0 \leqslant x_1 \wedge p_0(x_1))$.

Fort every natural number n let $\underline{n - functor} = \underline{functor} \cap F_0(L(\sigma \cup \{p_i : i < n\})) - \bigcup_{m<n} \underline{m - functor}$. In the case of $S4$ $\underline{0 - functor} = \emptyset$, $\underline{1 - functor} = \{H_\neg, H_\Box, H_\Diamond\}$, $\underline{2 - functor} = \{H_\wedge, H_\vee, H_\to, H_\leftrightarrow\}$ and for $n > 2$ $\underline{n - functor} = \emptyset$. For every $H \in \underline{functor}$ introduce a new symbol f_H which will act as the propositional functor associated with H. In the case of $S4$ let $f_{H_\neg} = \neg$, $f_{H_\wedge} = \wedge$, $f_{H_\vee} = \vee$, $f_{H_\to} = \to$, $F_{H_\leftrightarrow} = \leftrightarrow$, $f_{H_\Box} = \Box$ and $f_{H_\Diamond} = \Diamond$. Now we are ready to define the language of the propositional logic defined by the semantics $S = (\underline{K}, \underline{functor})$. The set $\underline{expr}(S)$ of all expressions is defined to be the least set such that

1) $\{p_i : i \in \omega\} \subseteq \underline{expr}(S)$
2) for every $H \in \underline{n - functor}$, $\alpha_0, \ldots, \alpha_{n-1} \in \underline{expr}(S)$

$$f_H \alpha_0, \ldots, \alpha_{n-1} \in \underline{expr}(S).$$

Next we define a function $\underline{meta} : \underline{expr}(S) \to F_0(L(\sigma \cup \{p_i : i < \omega\}))$. The formula $\underline{meta} - \alpha$ is defined by induction on the form of α. Let $\underline{meta} - p_i = p_i(x_0)$, $\underline{meta} - f_H\alpha_0 \ldots \alpha_{n-1} = H(p_i/\underline{meta} - \alpha_i)$. Here $H(p_i/\underline{meta} - \alpha_i)$ is the formula resulting from H by replacing each subformula of the form $p_i(t)$ where t is some term of $L(\sigma)$ by $\underline{meta} - \alpha_i(t)$. It is assumed that no confusion of variables is caused by these substitutions. Otherwise change the names of bound variables in $\underline{meta} - \alpha_i(x_0)$.

We have e.g. $\underline{meta} - \neg\Box\neg p_0 = \neg\forall x_1(x_0 \leqslant x_1 \to \neg p_0(x_1))$ in the case of $S4$.

Now we are ready to define for every $\underline{A} \in \underline{K}$, $a \in |\underline{A}|$, $\alpha \in \underline{expr}(S)$ and for every interpretation $\Vdash$ of the relational symbols p_i $(i < \omega)$ as subsets of $|\underline{A}|$

$$a \Vdash \alpha \text{ if and only if } (\underline{A}, \Vdash, a) \models \underline{meta} - \alpha(a).$$

α is said to be a tautology $(\alpha \in \underline{taut}(S))$ if $a \Vdash \alpha$ is true for every $a \in |\underline{A}|$ where $\underline{A} \in \underline{K}$ and for every interpretation $\Vdash$. By the same method we can obtain the usual Kripke semantics for several modal and tense logics. Now let $\underline{expr}$ be an arbitrary free algebra with countably many generators and let $\underline{taut}$ be a subset of $\underline{expr}$. The Kripke semantics $S = (\underline{K}, \underline{functor})$ is said to be adequate for the logic $(\underline{expr}, \underline{taut})$ if there is an isomorphism of $\underline{expr}$ onto $(\underline{expr}(S), f_H)_{H \in \underline{functor}}$ mapping $\underline{taut}$ onto $\underline{taut}(S)$. The proof of the following theorem can be found in [1].

Theorem. *The following conditions are equivalent.*

1) *There is a Kripke semantics adequate for* $(\underline{expr}, \underline{taut})$
2) *There is a logical matrix* $(M, M^*, \ldots)$ *which is adequate for* $(\underline{expr}, \underline{taut})$ *and such that* $M \neq M^*$ *and* M^* *has exactly one element.*

The presented method can be extended to non-classical predicate calculi. It makes it possible to employ the methods of classical model theory in the investigation of non-classical logics.

References

[1] B. Dahn, *Generalized Kripke models*, forthcoming in **Bull. Acad. Polon. des Sc.**, Sér. Sc. Mathem., Astronom., Phys.

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