

Andrzej Wroński

ON CARDINALITY OF MATRICES STRONGLY ADEQUATE FOR THE INTUITIONISTIC PROPOSITIONAL LOGIC

Gödel [2] stated that there is no finite matrix adequate for the intuitionistic propositional logic (*INT*). However, a denumerable adequate matrix was found by Jaśkowski [5]. In this paper it is shown that no denumerable matrix is strongly adequate for *INT* which was previously conjectured by prof. R. Suszko.

Let $\mathcal{F} = \langle F, \wedge, \vee, \rightarrow, \neg \rangle$ be the free algebra in the class of all algebras of the similarity type $\langle 2, 2, 2, 1 \rangle$ free-generated by a denumerably infinite set $V \subseteq F$. The elements of F are called formulas and denoted by α, β, \dots , the elements of V are called variables and denoted by x, y, \dots . The familiar abbreviation $\alpha \leftrightarrow \beta$ is used for $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. The symbol Sb denotes the consequence operation in F determined by the substitution rule, and Cn denotes the consequence operation in F determined by the set of theorems of *INT* (see [3]) and the detachment rule. If $X \subseteq F$ then \equiv_X denotes the congruence relation of \mathcal{F} given by the condition: $\alpha \equiv_X \beta$ iff $\alpha \leftrightarrow \beta \in Cn(X)$. By an intermediate logic we mean a set of formulas Δ such that $\Delta = Cn(Sb(\Delta)) \neq F$. The consequence operation of the intermediate logic Δ is denoted by Cn_Δ ($Cn_\Delta(X) = Cn(\Delta \cup X)$).

Any pair $\langle \mathcal{M}, D \rangle$ where \mathcal{M} is an algebra similar to \mathcal{F} and D is a subset of the domain of \mathcal{M} is called a matrix. $C_{\langle \mathcal{M}, D \rangle}$ denotes the consequence operation of the matrix $\langle \mathcal{M}, D \rangle$ ($\alpha \in C_{\langle \mathcal{M}, D \rangle}(X)$ iff $h(X) \subseteq D$ implies that $h(\alpha) \in D$ for every homomorphism $h : \mathcal{F} \mapsto \mathcal{M}$). For the sake of simplicity we use the symbol $E\langle \mathcal{M}, D \rangle$ instead of $C_{\langle \mathcal{M}, D \rangle}(\emptyset)$ for denoting the content of the matrix $\langle \mathcal{M}, D \rangle$. A matrix $\langle \mathcal{M}, D \rangle$ is said to be adequate (strongly adequate) for an intermediate logic Δ iff $E\langle \mathcal{M}, D \rangle = \Delta$ ($C_{\langle \mathcal{M}, D \rangle} = Cn_\Delta$). By a general theorem of Łoś, Suszko [6] (see also corrections in Wójcicki

[10]) we know that every intermediate logic has a strongly adequate matrix of the power not exceeding 2^{\aleph_0} . The following lemma shows that investigating matrices strongly adequate for intermediate logics we can confine ourselves to examining only so called pseudo-Boolean algebras (see [8]).

LEMMA 1. *If Δ is an intermediate logic and $\langle \mathcal{M}, D \rangle$ a matrix strongly adequate for Δ then the following conditions hold:*

- (i) *The relation \equiv_D such that $a \equiv_D b$ iff $a \leftrightarrow_{\mathcal{M}} b \in D$ is a congruence of the algebra \mathcal{M} ;*
- (ii) *\mathcal{M}/\equiv_D is a pseudo-Boolean algebra having D as unit-element;*
- (iii) *$\langle \mathcal{M}/\equiv_D, D/\equiv_D \rangle$ is a matrix strongly adequate for Δ .*

K denotes the class of all pseudo-Boolean algebras. The German capitals: $\mathcal{A}, \mathcal{B}, \dots$ denote algebras from K and the corresponding Latin capitals: A, B, \dots their domains. Every algebra $\mathcal{A} \in K$ in a natural way can be considered as a matrix $\langle \mathcal{A}, \{1_{\mathcal{A}}\} \rangle$ where $1_{\mathcal{A}}$ denotes the unit-element of \mathcal{A} . Thus, for the sake of simplicity, the symbols $C_{\mathcal{A}}$ and $E_{\mathcal{A}}$ will be used to abbreviate $C_{\langle \mathcal{A}, \{1_{\mathcal{A}}\} \rangle}$ and $E_{\langle \mathcal{A}, \{1_{\mathcal{A}}\} \rangle}$ respectively. \mathcal{A} is said to be strongly compact (see [8]) iff there exists the greatest element in $A - \{1_{\mathcal{A}}\}$ with respect to lattice ordering $\leq_{\mathcal{A}}$ of the algebra \mathcal{A} . Such an element – if it exists – will be denoted by $\star_{\mathcal{A}}$. The symbol K_0 denotes the class of all denumerable strongly compact algebras from K . With an intermediate logic Δ one can associate in an one-to-one manner an equational class of algebras $K(\Delta) = \{\mathcal{A} : \mathcal{A} \in K, \Delta \subseteq E_{\mathcal{A}}\}$ and also in such a manner a class of denumerable strongly compact algebras $K_0(\Delta) = K(\Delta) \cap K_0$. In the sequel it will be convenient to have the symbol $\mathcal{A} \oplus$ for denoting the result of applying Jaśkowski's Γ -operation (see [5]) to the algebra \mathcal{A} and the symbol \equiv_a for the congruence relation in \mathcal{A} determined by an element $a \in A$ ($b \equiv_a c$ iff $a \leq_{\mathcal{A}} b \leftrightarrow_{\mathcal{A}} c$).

LEMMA 2. *For every $\mathcal{A} \in K$ the following conditions hold:*

- (i) *$\mathcal{A} \oplus$ is strongly compact;*
- (ii) *$\mathcal{A} \oplus, \mathcal{B} \oplus$ are isomorphic iff so are \mathcal{A} and \mathcal{B} ;*
- (iii) *If $G \subseteq A$ generates \mathcal{A} then $G \cup \{\star_{\mathcal{A} \oplus}\}$ generates $\mathcal{A} \oplus$.*

Suppose that for each denumerable algebra $\mathcal{A} \in K$ we are given a fixed one-to-one mapping $f_{\mathcal{A}} : A \mapsto V$ (for the sake of simplicity the variable $f_{\mathcal{A}}(a)$ will be denoted by z_a). Generalizing an idea of Jankov [4] we define the diagram of a denumerable $\mathcal{A} \in K$ putting:

$$\begin{aligned} DG(\mathcal{A}) &= \{(z_a \wedge z_b) \leftrightarrow z_{a \wedge_{\mathcal{A}} b} : a, b \in A\} \cup \\ &\cup \{(z_a \vee z_b) \leftrightarrow z_{a \vee_{\mathcal{A}} b} : a, b \in A\} \cup \\ &\cup \{\neg z_a \leftrightarrow z_{\neg_{\mathcal{A}} a} : a \in A\} \end{aligned}$$

LEMMA 3. (comp. [4]). *For every $\mathcal{A} \in K_0$ and $\mathcal{L} \in K$ the following conditions are equivalent:*

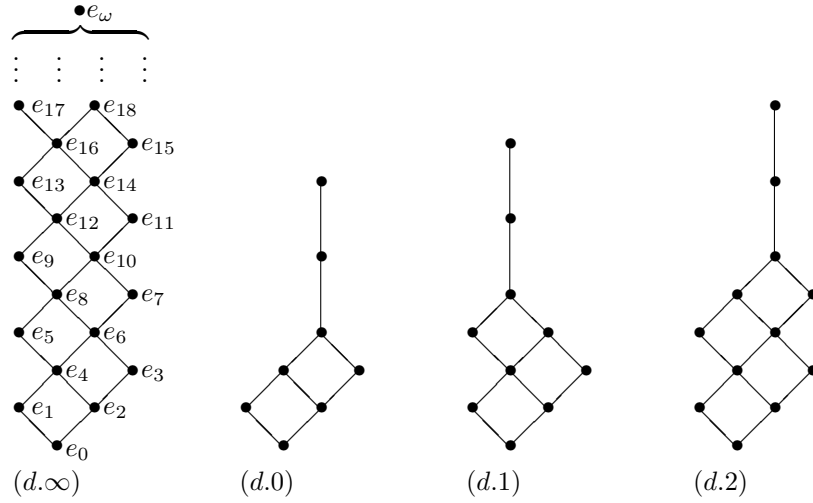
- (i) \mathcal{A} is embeddable into \mathcal{L} ;
- (ii) $z_{\star_{\mathcal{A}}} \notin C_{\mathcal{L}}(DG(\mathcal{A}))$.

An immediate consequence of Lemma 3 is the following useful criterion of the strong adequacy:

THEOREM 1. *If Δ is an intermediate logic and $\mathcal{L} \in K$ then the following conditions are equivalent:*

- (i) \mathcal{B} is strongly adequate for Δ ;
- (ii) $\mathcal{B} \in K(\Delta)$ and every algebra from $K_0(\Delta)$ is embeddable into \mathcal{B} .

Let φ_i ($i = 0, 1, \dots, \omega$) be the formulas defined as follows: $\varphi_0 = (y \wedge \neg y)$, $\varphi_1 = (\neg y)$, $\varphi_2 = (y)$, $\varphi_3 = (\neg \neg y)$, $\varphi_4 = (y \vee \neg y)$, $\varphi_{2i+5} = (\varphi_{2i+3} \rightarrow \varphi_{2i+2})$, $\varphi_{2i+6} = (\varphi_{2i+3} \vee \varphi_{2i+1})$, $\varphi_{\omega} = (y \rightarrow y)$. Rieger [9] and then (11 years later) Nischimura [7] proved that the equivalence classes of the formulas φ_i , $i = 0, 1, \dots, \omega$ under the congruence \equiv_{INT} are pairwise distinct and form a subalgebra of the quotient algebra $\mathcal{F} / \equiv_{INT}$. Obviously such a subalgebra is free in K with the one-element free-generating set. To the honour of the first finder let us denote it by \mathcal{R} and illustrate by the following picture ($d.\infty$) (e_i denotes the equivalence class of φ_i):



Putting $\vartheta_i \vartheta_i = (\mathcal{R} / \equiv_{e_{2i+7}}) \oplus$ for $i = 0, 1, \dots$ one obtains the sequence of algebras examined in Gerčiu, Kuznecov [1]. The pictures: (d.0), (d.1) and (d.2) visualize the lattice orderings of $\vartheta_0, \vartheta_1, \vartheta_2$ respectively. For every algebra ϑ_i we have the corresponding formula $\delta_i = \varphi_{2i+7} \rightarrow (x \vee (x \rightarrow \varphi_{2i+6}))$ (the two variables x and y occurring in δ_i are supposed to be distinct) and for every set of natural numbers $I \subseteq \omega$ the corresponding intermediate logic $L(I) = Cn(Sb(\delta_i : i \in I))$. Let us quote the following lemma due to Gerčiu, Kuznecov [1]:

LEMMA 4. (see [1]).

- (i) If $i \neq j$ then there is no embedding of ϑ_i into a quotient algebra of ϑ_j ;
- (ii) for every $\sqsubset \in K$, $\delta_i \in E \sqsubset$ iff there is no embedding of ϑ_i into a quotient algebra of \sqsubset ;
- (iii) $\delta_i \notin L(\omega - (i))$;
- (iv) $L(I) = L(J)$ iff $I = J$.

Let $\mathcal{F}_2 = \langle F_2, \wedge, \vee, \rightarrow, \neg \rangle$ be the subalgebra of \mathcal{F} generated by the variables: x and y . As a simple corollary of Lemma 4 (iv) we have the following:

LEMMA 5. *The algebras: $\mathcal{F}_2/\equiv_{L(I)}$, $\mathcal{F}_2/\equiv_{L(J)}$ are isomorphic iff $I = J$.*

THEOREM 2. *The number of non-isomorphic algebras in K_0 having a three-element generating set is 2^{\aleph_0} .*

Combining Theorem 1, 2 and Lemma 1 we obtain the following:

THEOREM 3. *There is no denumerable matrix strongly adequate for INT.*

References

- [1] V. Ja. Gerčiu, A. V. Kuznecov, *On finitely axiomatizable superintuitionistic logics*, **Dokl. Akad. Nauk SSSR** 159 (1970), pp. 1263–1266 = **Soviet. Math. Dokl.** 11 (1970), pp. 1654–1658.
- [2] K. Gödel, *Zum intuitionistischen Aussagenkalkül*, **Akad. Wiss. Anzeiger** 69 (1932), pp. 65–66.
- [3] K. Heyting, *Die formalen Regeln der intuitionistischen Logik*, **Prenss. Akad. Wiss. Phys.-Math.Kl.** (1930), pp. 42–56.
- [4] V. A. Jankov, *Relationship between deducibility in the intuitionistic propositional calculus and finite implicational structures*, **Dokl. Akad. Nauk SSSR** 151 (1963), pp. 1293–1294 = **Soviet Math. Dokl.** 4 (1963), pp. 1203–1204.
- [5] S. Jaśkowski, *Recherches sur le système de la logique intuitionistic*, **Actualities sci. et industrielles** 393 (1936), pp. 58–61.
- [6] J. Łoś, R. Suszko, *Remarks on sentential logics*, **Indag. Math.** 20 (1958), pp. 177–183.
- [7] I. Nischimura, *On formulas of one variable in intuitionistic propositional calculus*, **J.S.L.** 25 (1960), pp. 327–331.
- [8] H. Rasiowa, R. Sikorski, **The mathematics of metamathematics**, Warsaw, PWN 1968.
- [9] L. Rieger, *On the lattice theory of Brouwerian propositional logic*, **Acta facultatis rerum naturalium Universitas Carolinae** 189 (1949), pp. 1–40.
- [10] R. Wójcicki, *Some remarks on the consequence operation in sentential logics*, **Fund. Math.** 68 (1970), pp. 269–279.